

The \hat{A} -genus as a projective volume form on the derived loop space

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Abstract

We consider the one-dimensional Chern-Simons theory given by an L_∞ algebra \mathfrak{g} . The quantization of this theory produces a projective volume form on the derived loop space of $B\mathfrak{g}$. The resulting integration can be identified with integration of differential forms on $B\mathfrak{g}$ twisted by $\hat{A}(B\mathfrak{g})$. We further analyze the cases where \mathfrak{g} encodes the smooth/holomorphic geometry of a manifold. We extensively use an approach to derived geometry given by L_∞ spaces and differential graded manifolds.

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1 Introduction

In previous work of Owen Gwilliam and the author [GGb], a one-dimensional Chern-Simons theory was described as a BV theory for any L_∞ algebra. These Chern-Simons theories were direct analogs of Costello's holomorphic Chern-Simons theory [Cosa]. It was shown that one can encode the smooth geometry of a manifold as an L_∞ algebra and that the resulting Chern-Simons theory is the infinite volume limit of the one-dimensional sigma model.

The formulation and quantization of such theories is described in the language of L_∞ spaces and differential graded manifolds. Given an L_∞ space (X, \mathfrak{g}) , there is an associated differential graded manifold which we denote by $B\mathfrak{g}$. Such spaces can be seen as one approach to derived geometry. Standard geometric constructions, such as (co)tangent bundles, have analogues in this setting. Further, under good conditions we can define nice mapping spaces. The *derived loop space* of an L_∞ space (X, \mathfrak{g}) is given by the space of maps from the differential graded manifold $(S^1, \Omega_{S^1}^*)$ to (X, \mathfrak{g}) . We denote the derived loop space by $\mathcal{L}B\mathfrak{g}$. In [GGb], we developed characteristic classes for L_∞ spaces by using the formalism of Atiyah classes.

For the L_∞ algebra \mathfrak{g}_X which encodes the smooth geometry of a manifold X it is critical to remember the homotopy S^1 action on the derived loop space of (X, \mathfrak{g}_X) . By such considerations, we relate a certain characteristic class $\hat{A}_u(B\mathfrak{g}_X)$ to the classical $\hat{A}(X)$.

With Gwilliam, we showed that one-dimensional Chern-Simons can be quantized (in fact, at one loop) and thus defines a quantum field theory. An explicit computation proved that at the *partition function* of the quantized theory encoded the characteristic class $\hat{A}(B\mathfrak{g})$. As a corollary, we wrote down the complexes of global classical and quantum observables. The structure of the local quantum observables is described in a follow up paper [GGa].

In this paper, we explain how our quantization constructs a volume form on $\mathcal{L}B\mathfrak{g}$ and allows us to define an integration map for functions on $\mathcal{L}B\mathfrak{g}$. We can explicitly realize this integration as

$$\mathcal{O}(\mathcal{L}B\mathfrak{g}) \ni f \mapsto \int_{T[-1]B\mathfrak{g}} f \hat{A}(B\mathfrak{g}) dVol_0 \in \mathbb{R},$$

where $dVol_0$ is a canonical volume form on $T[-1]B\mathfrak{g}$. In the case that \mathfrak{g} encodes the holomorphic structure of a complex manifold X , Costello has shown that integration against $dVol_0$ is just integration of differential forms over X . When \mathfrak{g} encodes the smooth structure of a manifold the situation is more delicate as we must consider a homotopy S^1 action; this is discussed in the last section of the present work.

As a note to the reader, the manuscript proceeds in three movements. Sections 2,3, and 4 are a meditation on the notion of L_∞ space and their relationship to differential graded manifolds as described by Costello in [Cosa]. Section 5 is an unusually dense segue heavily influenced by the work of Calaque and Van den Bergh [CVdB10]. Sections 6,7, and 8 recall the critical themes and previous results of Gwilliam and the author. Finally, Sections 9, 10, 11, and 12 describe the resulting projective volume forms and contain most of the novel content of the present manuscript.

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2 L_∞ spaces

An L_∞ space is a ringed space with a structure sheaf a sheaf L_∞ algebras, where an L_∞ algebra is the homotopical enhancement of a differential graded Lie algebra.

2.1 L_∞ algebras

Let A differential commutative graded algebra and $I \subset A$ a nilpotent ideal. Let A^\sharp denote the underlying graded algebra i.e. we forget the differential.

Definition 2.1. Let M be a module over A . M is locally free if the localization M_P is free over R_P for each prime ideal $P \subset R$.

One should think of a locally free finitely generated module as the algebraic notion of vector bundle (this can be made precise provided that the base ring A is nice enough).

Definition 2.2. A curved L_∞ algebra over A consists of a locally free finitely generated graded A^\sharp -module V , together with a cohomological degree 1 and square zero derivation

$$d : \widehat{\text{Sym}}(V^\vee[-1]) \rightarrow \widehat{\text{Sym}}(V^\vee[-1])$$

where V^\vee is the A^\sharp -linear dual and the completed symmetric algebra is also over A^\sharp . There are two additional requirements on the derivation d :

1. The derivation d makes $\widehat{\text{Sym}}(V^\vee[-1])$ into a dga over the dga A ;
2. Reduced modulo the nilpotent ideal $I \subset A$, the derivation d preserves the ideal in $\widehat{\text{Sym}}(V^\vee[-1])$ generated by V .

Note that our dualizing convention is such that $V^\vee[-1] = V[1]^\vee$.

We can decompose the derivation d into its constituent pieces

$$d_n : V^\vee[-1] \rightarrow \text{Sym}^n(V^\vee[-1])$$

and after dualizing and shifting we obtain maps

$$l_n : \Lambda^n V[n-2] \rightarrow V.$$

The maps $\{l_n\}$ satisfy higher Jacobi relations [LM95]. In particular, if $l_n = 0$ for all $n \neq 2$, then V is just a graded Lie algebra. Similarly, if $l_n = 0$ for all $n \neq 1, 2$, then V is a differential graded Lie algebra. If $l_n = 0$ for $n \neq 1, 2, 3$ then l_3 is a contracting homotopy for the Jacobi relation, i.e.

$$\begin{aligned} & (-1)^{|x| \cdot |z|} l_2(l_2(x, y), z) + (-1)^{|y| \cdot |z|} l_2(l_2(z, x), y) + (-1)^{|x| \cdot |y|} l_2(l_2(y, z), x) = \\ & (-1)^{|x| \cdot |z| + 1} (l_1 l_3(x, y, z) + l_3(l_1 x, y, z) + (-1)^{|x|} l_3(x, l_1(y), z) + (-1)^{|x| + |y|} l_3(x, y, l_1 z)). \end{aligned}$$

If V is a L_∞ algebra over A , then $C^*(V)$ will denote the differential graded A -algebra $\widehat{\text{Sym}}(V^\vee[-1])$. Our convention will be that V is concentrated in non-negative degrees, so that $C^{>0}(V)$ is a (maximal) ideal of $C^*(V)$.

Remark 2.3. There is a Quillen equivalence between the categories of differential graded lie algebras and cocommutative coalgebras with coderivation, see [Qui69]. We can view this equivalence as an example of Koszul duality and extend it to the cofibrant replacements of the relevant operads (cocomm and Lie) in chain complexes. If the relevant chain complexes are dualizable then we dualize and obtain the definition as presented above.

2.2 L_∞ spaces

Let X be a manifold and consider the nilpotent ideal $\Omega_X^{>0} \subset \Omega_X^*$.

Definition 2.4. An L_∞ space is a manifold X equipped with a sheaf \mathfrak{g} of L_∞ algebras over Ω_X^* which is locally free of finite total rank (as graded Ω_X^\sharp -modules).

Definition 2.5. Given an L_∞ space (X, \mathfrak{g}) , the reduced structure sheaf $\mathfrak{g}_{\text{red}}$ is defined by

$$\mathfrak{g}_{\text{red}} = \mathfrak{g} / \Omega_X^{>0}.$$

One should think of the reduced structure sheaf as something like the dual to the cotangent complex and hence a measure of the “niceness” of the L_∞ space (X, \mathfrak{g}) .

Proposition 2.6. Given an L_∞ space (X, \mathfrak{g}) , the reduced structure sheaf $\mathfrak{g}_{\text{red}}$ has no curving i.e. $l_1^2 = 0$.

Proof. From the L_∞ relations we know that $l_1^2 = l_0$. Now $l_0 : \mathbb{C} \rightarrow V$ i.e. l_0 is just an element of V which is dual to the map $d_0 : V^\vee[-1] \rightarrow \mathbb{C}$. The condition that reduced modulo the nilpotent ideal I the derivation d preserves the ideal generated by V implies that $l_0 \in V \otimes_A I$. Therefore reduced modulo I , $l_0 = 0$. \square

2.3 Morphisms of L_∞ spaces

A map $\alpha : \mathfrak{g} \rightarrow \mathfrak{h}$ of L_∞ algebras is given by a sequence of linear maps

$$\text{Sym}^n(\mathfrak{g}[1]) \rightarrow \mathfrak{h}$$

of degree 1 satisfying certain quadratic identities. If \mathfrak{h} is finite dimensional, then the map α is exactly a map of differential graded algebras

$$C^*(\mathfrak{h}) \rightarrow C^*(\mathfrak{g})$$

which takes the maximal ideal $C^{>0}(\mathfrak{h})$ to the maximal ideal $C^{>0}(\mathfrak{g})$. Alternatively, we can view α as an element $\alpha \in C^*(\mathfrak{g}) \otimes \mathfrak{h}$ satisfying the Maurer-Cartan equation

$$d\alpha + \sum_{n \geq 1} \frac{1}{n!} l_n(\alpha, \dots, \alpha) = 0$$

and which vanishes modulo the maximal ideal $C^{>0}(\mathfrak{g})$ (see for instance [KS]). This definition continues to make sense for curved L_∞ algebras over a base ring A with nilpotent ideal I by asking that

$$\sum_{n \geq 0} \frac{1}{n!} l_n(\alpha, \dots, \alpha) = 0$$

and that α vanishes modulo the ideal generated by $C^{>0}(\mathfrak{g})$ and I .

Let (X, \mathfrak{g}) be an L_∞ space and Y is a smooth manifold. Given a smooth map $\phi : Y \rightarrow X$ we have the pull back L_∞ algebra over Ω_Y^* given by

$$\phi^* \mathfrak{g}_X \stackrel{def}{=} \phi^{-1} \mathfrak{g}_X \otimes_{\Omega_X^*} \Omega_Y^*.$$

Here $\phi^{-1} \mathfrak{g}_X$ denotes the sheaf pull back.

Definition 2.7. Let (X, \mathfrak{g}_X) and (Y, \mathfrak{g}_Y) be L_∞ spaces, then a morphism

$$\phi : (Y, \mathfrak{g}_Y) \rightarrow (X, \mathfrak{g}_X)$$

is given by a smooth map $\phi : Y \rightarrow X$ and a map of curved L_∞ algebras over Ω_Y^*

$$\mathfrak{g}_Y \rightarrow \phi^* \mathfrak{g}_X.$$

We also have the notion of equivalence for L_∞ algebras as cochain homotopy equivalence of the reduced algebras which leads to a definition of equivalence on the space level.

Definition 2.8. An L_∞ map $\phi : (Y, \mathfrak{g}_Y) \rightarrow (X, \mathfrak{g}_X)$ is an equivalence of L_∞ spaces if the underlying map $Y \rightarrow X$ is a diffeomorphism and if the map of curved L_∞ algebras $\mathfrak{g}_Y \rightarrow \phi^* \mathfrak{g}_X$ is an equivalence i.e. the map

$$(\mathfrak{g}_Y)_{red} \rightarrow (\phi^* \mathfrak{g}_X)_{red}$$

is a cochain homotopy equivalence of sheaves C_Y^∞ modules.

This notion of equivalence is quite strong. If \mathfrak{g} and \mathfrak{h} are equivalent as curved L_∞ algebras over Ω_Y^* , then $C^*(\mathfrak{g})$ and $C^*(\mathfrak{h})$ are homotopy equivalent, but the converse is not necessarily true. Note that $C^*(\mathfrak{g})$ (and similarly for $C^*(\mathfrak{h})$) is filtered by powers of the ideal generated by $\Omega_Y^{>0}$ and \mathfrak{g}^\vee , the associated graded is $\text{Sym}(\mathfrak{g}_{red}^\vee[-1])$. The definition of equivalence implies that we have an equivalence at the first page of the associated spectral sequences. One reason why this stronger definition is desirable is if we consider L_∞ spaces with underlying manifold a point, then there are L_∞ algebras (even just Lie algebras) that have quasi-isomorphic Chevalley-Eilenberg complexes yet that are quite different; for instance, it is well known that $H^*(\mathfrak{sl}_2(\mathbb{C}))$ is an exterior algebra on one generator in each degree $3n$ for $n \geq 0$, but the rank 1 free Abelian lie algebra concentrated in degree -2 has the same cohomology.

Remark 2.9. Note that the category of L_∞ spaces can be simplicially enriched. The n -simplices of the set of maps (Y, \mathfrak{g}_Y) to (X, \mathfrak{g}_X) are smooth maps $\phi : Y \rightarrow X$ and a map of curved L_∞ algebras over Ω_Y^*

$$\mathfrak{g}_Y \rightarrow \phi^* \mathfrak{g}_X \otimes \Omega_{\Delta^n}^*$$

where the right hand side makes sense as L_∞ algebras are tensored over cdgas. One advantage of this perspective is that it allows us to define families of L_∞ structures and a natural notion of homotopy. It is non trivial, yet true (as shown in [Cosa]) that the simplicial structure is compatible with the definition of equivalence in L_∞ spaces. i.e if $\phi : (Y, \mathfrak{g}_Y) \rightarrow (X, \mathfrak{g}_X)$ is an equivalence then for any other L_∞ space (Z, \mathfrak{g}_Z) the induced maps of of simplicial sets

$$\begin{aligned} \text{Maps}((Z, \mathfrak{g}_Z), (Y, \mathfrak{g}_Y)) &\rightarrow \text{Maps}((Z, \mathfrak{g}_Z), (X, \mathfrak{g}_X)) \\ \text{Maps}((X, \mathfrak{g}_X), (Z, \mathfrak{g}_Z)) &\rightarrow \text{Maps}((Y, \mathfrak{g}_Y), (Z, \mathfrak{g}_Z)) \end{aligned}$$

are weak homotopy equivalences.

3 dg-manifolds

Definition 3.1. A *naive dg-manifold* is a pair (X, \mathcal{O}_X) with X a smooth manifold and \mathcal{O}_X a sheaf of commutative differential graded algebras (cdgas) such that the underlying graded algebra is locally modeled on $C_X^\infty \otimes \widetilde{\text{Sym}}(V)$ for V a finite dimensional vector space (over \mathbb{R} or \mathbb{C}).

We view dg-manifolds and L_∞ spaces as one approach to derived geometry.

Example 3.2.

- (1) Let $M, N \subset X$ be submanifolds, then the *derived intersection* is a dg-manifold given by

$$M \overset{\mathbb{L}}{\bigcap} N = (X, C_M^\infty \otimes_{C_X^\infty}^\mathbb{L} C_N^\infty)^1.$$

- (1b) Let $f : X \rightarrow \mathbb{R}$ be a smooth function then the *derived critical locus* is a dg-manifold $(X, \mathcal{O}_{\text{dcrit}(f)})$ with

$$\mathcal{O}_{\text{dcrit}(f)} = \cdots \rightarrow \Lambda^2 T_X[2] \xrightarrow{i_{df}} T_X[1] \xrightarrow{i_{df}} C_X^\infty.$$

- (2) For X a smooth manifold, the *de Rham stack* X_Ω is a dg-manifold given as (X, Ω_X^*) , where Ω_X^* is the de Rham complex of X .

Note that *the* derived intersection is only defined up to equivalence and we should freely resolve both C_X^∞ modules in order for the example to fit our definitions. The following lemma is not critical to what follows, but is meant to elucidate the nature of dg-manifolds.

Lemma 3.3. *The derived critical locus, $\text{dcrit}(f)$, is a derived intersection.*

Proof. For simplicity, let us assume that $X = \mathbb{R}\langle x \rangle$ is one dimensional. We write $\mathcal{O}_{T^*M} = C_X^\infty[dx]$, so the functions on the zero section are given by the module $\mathcal{O}_0 = C_X^\infty[dx]/(dx)$. We resolve functions on the graph of f via its Koszul complex:

$$K = 0 \rightarrow C_X^\infty[dx, \partial_x] \xrightarrow{\partial_x \mapsto dx - \partial f / \partial x} C_X^\infty[dx].$$

¹Note that here and throughout $\otimes^\mathbb{L}$ indicates the derived tensor product.

Then we have

$$\mathcal{O}_0 \otimes_{\mathcal{O}_{T^*M}} K = C_X^\infty[\partial_x] \xrightarrow{\partial_x \mapsto -\partial f / \partial x} C_X^\infty$$

as desired. This construction holds locally for an arbitrary one-dimensional manifold. The generalization to higher dimensions is straight forward. \square

Remark 3.4. Again by using the Koszul complex for the (locally) defining regular sequence (see [Eis95] or [KS]) we have the following examples.

- There is a functor F_{Aff} from the category of smooth² affine schemes to naive dg-manifolds;
- There is a functor F_{LCI} from the category of smooth local complete intersections to naive dg-manifolds.

Throughout the sequel we actually use a slightly different definition of *dg-manifold*. One should think of this second definition as a naive dg-manifold over X_Ω .

Definition 3.5. A *dg-manifold* is a pair (X, \mathcal{O}_X) with X a smooth manifold and \mathcal{O}_X a sheaf of commutative differential graded Ω_X algebras such that the underlying graded algebra is locally modeled on $\Omega_X^* \otimes \widehat{\text{Sym}}(V)$ for V a finite dimensional vector space. We also ask that \mathcal{O}_X is equipped with a map $\mathcal{O}_X \rightarrow C_X^\infty$ whose kernel $I \subset \mathcal{O}_X$ is a sheaf of nilpotent ideals.

Note that this definition differs from [Cosa]. One upshot of the current definition is that given an L_∞ space (X, \mathfrak{g}) there is a corresponding dg-manifold $(X, C^*(\mathfrak{g}))$. Whereas for Costello, $(X, C^*(\mathfrak{g}))$ is only a pro-object.

Definition 3.6. A morphism of dg-manifolds from (Y, \mathcal{O}_Y) to (X, \mathcal{O}_X) is a pair of maps (f, f^\sharp) with $f : Y \rightarrow X$ a smooth map and $f^\sharp : f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y$ a map of sheaves of $f^{-1}\Omega_X^*$ -algebras which covers the map $f^{-1}C_X^\infty \rightarrow C_Y^\infty$.

Similar to above, note that if f is a diffeomorphism then the sheaves $f^{-1}\mathcal{O}_X$ and \mathcal{O}_Y are filtered by the powers of the nilpotent ideal which is the kernel of the map $\mathcal{O}_Y \rightarrow C_Y^\infty$.

Definition 3.7. A map (f, f^\sharp) from (Y, \mathcal{O}_Y) to (X, \mathcal{O}_X) is an equivalence if the map $f : Y \rightarrow X$ is a diffeomorphism and the map f^\sharp induces a quasi-isomorphism at the level of associated graded algebras.

3.1 Comparing L_∞ spaces and dg-manifolds

Lemma 3.8. *The Chevalley-Eilenberg complex defines a full and faithful functor*

$$C^*(-) : L_\infty \text{ spaces} \rightarrow \text{dg-manifolds}$$

which takes equivalences to equivalences.

²Here *smooth* is meant in the differential geometric sense and not the more general algebraic geometric sense.

Proof. Let $\phi : (Y, \mathfrak{g}_Y) \rightarrow (X, \mathfrak{g}_X)$ be a map of L_∞ spaces, so we have the underlying smooth map $\phi_0 : Y \rightarrow X$ and a map Ω_Y^* -algebras

$$\phi_1 : C^*(\phi_0^* \mathfrak{g}_X) \rightarrow C^*(\mathfrak{g}_Y).$$

Now as $\phi_0^{-1} \Omega_X^*$ -algebras, we have

$$\begin{aligned} C^*(\phi_0^* \mathfrak{g}_X) &= C^*\left(\phi_0^{-1} \mathfrak{g}_X \otimes_{\phi_0^{-1} \Omega_X^*} \Omega_Y^*\right) \\ &= \widehat{\text{Sym}}_{\Omega_Y^*} \left(\left(\phi_0^{-1} \mathfrak{g}_X \otimes_{\phi_0^{-1} \Omega_X^*} \Omega_Y^* \right)^\vee [-1] \right) \\ &\cong \widehat{\text{Sym}}_{\phi_0^{-1} \Omega_X^*} \left(\left(\phi_0^{-1} \mathfrak{g}_X \right)^\vee [-1] \right) \\ &= \phi_0^{-1} C^*(\mathfrak{g}_X), \end{aligned}$$

so we have a map

$$\phi_1^\sharp : \phi_0^{-1} C^*(\mathfrak{g}_X) \rightarrow C^*(\mathfrak{g}_Y)$$

induced by ϕ_1 . What remains to check is that ϕ_1^\sharp covers the map on functions $\phi_0^{-1} C_X^\infty \rightarrow C_Y^\infty$ which follows from the fact that the map of L_∞ algebras ϕ_1 took the maximal ideal in $C^*(\phi_0^* \mathfrak{g}_X)$ to the maximal ideal in $C^*(\mathfrak{g}_Y)$. As we saw above equivalences of L_∞ spaces were defined so that they induced homotopy equivalences at the level of associated graded. That this functor is full and faithful follows from the adjointness of restriction and extension of scalars. \square

4 Geometric constructions on L_∞ spaces

Definition 4.1. Let (X, \mathfrak{g}) be an L_∞ space.

- A *vector bundle* V on (X, \mathfrak{g}) is a locally free sheaf of Ω_X modules such that $V \oplus \mathfrak{g}$ has the structure of a curved L_∞ algebra over Ω_X satisfying
 - The maps $\mathfrak{g} \hookrightarrow V \oplus \mathfrak{g}$ and $V \oplus \mathfrak{g} \rightarrow \mathfrak{g}$ are maps of L_∞ algebras;
 - The Taylor coefficients l_n vanish on tensors containing two or more sections of V .
- The *sheaf of sections* of V is given by $C^*(\mathfrak{g}, V[1])$ as a sheaf of dg modules over $C^*(\mathfrak{g})$.

The L_∞ space $(X, V \oplus \mathfrak{g})$ is the total space of the vector bundle given by $V[1]$ formally completed along the zero section.

4.1 (Co)Tangent bundle

Let V be a vector space (finite dimensional or topological) which we can think of as a dg-manifold with underlying manifold a point. We define functions on V by the (completed) symmetric algebra of the dual. Now functions on the tangent bundle TV are given by

$$\mathcal{O}(TV) = \mathcal{O}_V \otimes_{\mathcal{O}_V} \text{Sym}_{\mathcal{O}_V} (\text{Der}(\mathcal{O}_V)^\vee).$$

Hence, the tangent bundle $T(X, \mathfrak{g})$ is given by the \mathfrak{g} module $\mathfrak{g}[1]$. Vector fields are sections of the tangent bundle and hence as a sheaf are $C^*(\mathfrak{g}, \mathfrak{g}[1])$. Another way to see this is as follows. Consider any graded vector bundle $E \rightarrow X$, then for any $U \subset X$ open we have an identification of vector spaces ([CG] Lemma B.3.0.1)

$$\text{Der}(\mathcal{O}(\mathcal{E}(U))) \cong \mathcal{O}(\mathcal{E}(U)) \otimes \mathcal{E}(U).$$

This equivalence follows from the fact that any such derivation is determined by its value on the generators and hence is determined by a map

$$\mathcal{E}(U)^\vee \rightarrow \mathcal{O}(\mathcal{E}(U)).$$

We define the cotangent bundle $T^*(X, \mathfrak{g})$ to be the dual module to the tangent bundle i.e. $\mathfrak{g}^\vee[-1]$. A k -form on (X, \mathfrak{g}) is a section of the k th exterior power of the cotangent bundle, where

$$\Lambda^k T^*(X, \mathfrak{g}) = \Lambda^k(\mathfrak{g}^\vee[-1]) = \text{Sym}^k(\mathfrak{g}^\vee)[-k].$$

So a k -form is a section of the sheaf $C^*(\mathfrak{g}, \text{Sym}^k(\mathfrak{g}^\vee)[-k])$.

The total space of the tangent bundle is given by

$$T(X, \mathfrak{g}) = (X, \mathfrak{g} \oplus \mathfrak{g}),$$

while the total space of the cotangent bundle is given by

$$T^*(X, \mathfrak{g}) = (X, \mathfrak{g} \oplus \mathfrak{g}^\vee[-2]).$$

We also have the shifted version of the tangent and cotangent bundles. Of note, we have

$$T^*[-1](X, \mathfrak{g}) = (X, \mathfrak{g} \oplus \mathfrak{g}^\vee[-3]) \text{ and } T[-1](X, \mathfrak{g}) = (X, \mathfrak{g}[\epsilon])$$

where ϵ is a square zero parameter of degree 1.

4.2 Mapping spaces

We are interested in studying the space of maps from a dg-manifold (M, \mathcal{O}_M) to an L_∞ space (X, \mathfrak{g}) . One should think of this construction as constructing an internal hom on the category of generalized dg-manifolds (after embedding L_∞ spaces inside of dg-manifolds). The data of a single map is a smooth map $\phi : M \rightarrow X$ and a Maurer-Cartan element

$$\alpha \in \phi^* \mathfrak{g} \otimes_{\Omega_M^*} \mathcal{O}_M$$

which vanishes modulo the nilpotent ideal $I \subset \mathcal{O}_M$. The space of all maps is naturally a simplicial presheaf on the category of dg-manifolds which we denote $MC((M, \mathcal{O}_M), (X, \mathfrak{g}))$; it associates to a dg-manifold (N, \mathcal{O}_N) the simplicial set of maps from the dg-manifold $(N \times M, \mathcal{O}_N \boxtimes \mathcal{O}_M)$ to the L_∞ space (X, \mathfrak{g}) i.e. a smooth map and a Maurer-Cartan element. Now define the subsheaf

$$\widehat{MC}((M, \mathcal{O}_M), (X, \mathfrak{g})) \subset MC((M, \mathcal{O}_M), (X, \mathfrak{g}))$$

to be the subset of maps where the map on the underlying manifolds is constant.³

Under certain conditions, the functor $\widehat{MC}((M, \mathcal{O}_M), (X, \mathfrak{g}))$ is representable by an L_∞ space. The conditions are a bit technical and are presented as Proposition 5.0.1 of [Cosa]. The upshot is that if (X, \mathfrak{g}) is an L_∞ space such that $\mathfrak{g}_{\text{red}}$ has cohomology only in degrees ≥ 1 , then for any manifold M the mapping presheaf $\widehat{MC}((M, \Omega_M^*), (X, \mathfrak{g}))$ is represented by the L_∞ space $(X, \mathfrak{g} \otimes \Omega_M^*)$. In particular, consider the case where $M = S^1$, we call the resulting L_∞ space the *derived loop space* and will denote it

$$\mathcal{L}(X, \mathfrak{g}) = (X, \mathfrak{g} \otimes \Omega_{S^1}^*).$$

Proposition 4.2. *Let (X, \mathfrak{g}) be an L_∞ space such that the cohomology of $\mathfrak{g}_{\text{red}}$ is concentrated in degrees greater than 0. Then as L_∞ spaces we have*

$$\mathcal{L}(X, \mathfrak{g}) \cong T[-1](X, \mathfrak{g}).$$

Proof. Let $\mathcal{H}(S^1) \subset \Omega_{S^1}^*$ denote the harmonic forms, then

$$\mathcal{L}(X, \mathfrak{g}) = (X, \mathfrak{g} \otimes \Omega_{S^1}^*) \cong (X, \mathfrak{g} \otimes \mathcal{H}(S^1)).$$

By fixing a volume form on S^1 we have an isomorphism $\mathcal{H}(S^1) \cong \mathbb{C}[\epsilon]$, where ϵ is square zero of degree 1. Therefore we obtain an equivalence of L_∞ spaces

$$\mathcal{L}(X, \mathfrak{g}) \cong (X, \mathfrak{g} \otimes \mathcal{H}(S^1)) \cong (X, \mathfrak{g} \otimes \mathbb{C}[\epsilon]) \cong T[-1](X, \mathfrak{g}).$$

□

5 The Atiyah class

In order to define characteristic classes in setting of L_∞ spaces we recall the Atiyah class and some of its properties. Our presentation is based on the approach to the Atiyah class in the differential graded setting of Calaque and Van den Bergh [CVdB10]. We will elaborate on how these constructions appear in the geometry of manifolds in Section 7.2.

5.1 The definition

Let $R = (R^\#, d)$ be a commutative dga over a base ring k . The underlying graded algebra is denoted $R^\#$. We denote the Kähler differentials of R by Ω_R^1 and let $d_{dR} : R \rightarrow \Omega_R^1$ denote the universal derivation.

Definition 5.1. Let M be an R -module that is projective over $R^\#$. A *connection* on M is a k -linear map $\nabla : M \rightarrow M \otimes_R \Omega_R^1$ such that

$$\nabla(r \cdot m) = (d_{dR} r)m + (-1)^{|r|} r \nabla m,$$

for all $r \in R$ and $m \in M$.

³More precisely, $\widehat{MC}((M, \mathcal{O}_M), (X, \mathfrak{g}))(N, \mathcal{O}_N)$ is the sub-simplicial set consisting of those maps $N \times M \rightarrow X$ that factor through the projection $N \times M \rightarrow N$.

A connection may not be compatible with the differential d_M on M , and the Atiyah class is precisely the obstruction to compatibility between ∇ and the dg R -module structure on M .

Definition 5.2. The *Atiyah class* of ∇ is the class in $\Omega_R^1 \otimes_R \text{End}_R(M)$ given by

$$\text{At}(\nabla) = [\nabla, d] = \nabla \circ d_M - d_{\Omega_R^1 \otimes_R M} \circ \nabla.$$

This definition is quite abstract as stated, but it appears naturally in many contexts, notably in work by Kapranov [Kap99], Markarian [Mar09], Caldararu [CW10] [Că105], Ramadoss [Ram08] and Chen-Stiénon-Xu [CSX].

Remark 5.3. Atiyah [Ati57] originally introduced this construction to measure the obstruction to obtaining a holomorphic connection on a holomorphic bundle over a complex manifold. Let X be a complex manifold, $\pi : E \rightarrow X$ a holomorphic vector bundle, $\Omega_X^{0,*}$ the Dolbeault complex of X , and $(\Omega^{0,*}(E), \bar{\partial})$ the Dolbeault complex of the bundle. Let

$$\nabla : \Omega^{0,*}(E) \rightarrow \Omega_X^{1,*} \otimes_{\Omega_X^{0,*}} \Omega^{0,*}(E)$$

be a \mathbb{C} -linear map satisfying

$$\nabla(fs) = (\partial f)s + f\nabla s$$

for all $f \in \Omega_X^{0,*}$ and $s \in \Omega^{0,*}(E)$ (really it is enough to consider ∇ on $\Omega^{0,0}(E)$). The usual Atiyah class is $[\nabla, \bar{\partial}] \in \Omega^{1,1}(\text{End}(E))$. Notice that if this Atiyah class vanishes, then ∇ is clearly a holomorphic connection. On a compact Kähler manifold, Atiyah showed that traces of powers of the usual Atiyah class give the Chern classes of E .

5.2 Koszul duality and the Atiyah class

In the setting of L_∞ -algebras, we take the Chevalley-Eilenberg complex as *the definition* of the L_∞ structure, so it should be no surprise that there is a natural way to strip off the Taylor components from the Chevalley-Eilenberg complex. What we'll show in this section is that the tangent bundle to $B\mathfrak{g}^4$ has a natural connection and that by taking derivatives of its Atiyah class, we recover the brackets ℓ_n of the L_∞ -algebra \mathfrak{g} . This result is interesting from the point of view of deformation theory and Koszul duality: it explains how the Atiyah class fits into the process that constructs from a commutative dga \mathcal{A} the Koszul dual L_∞ -algebra $\mathfrak{g}_{\mathcal{A}}$.⁵

We will work with an arbitrary \mathfrak{g} -module M as it simplifies the formulas to distinguish between M and \mathfrak{g} (for the tangent bundle, M is another copy of \mathfrak{g} , which can be distracting). Consider the sections \mathcal{M} of this module as a sheaf over $B\mathfrak{g}$: it is the $C^*(\mathfrak{g})$ -module $C^*(\mathfrak{g}, M)$. Forgetting the differentials, we see there is a natural trivialization

$$C^\#(\mathfrak{g}, M) \cong C^\#(\mathfrak{g}) \otimes_k M,$$

⁴Here we $B\mathfrak{g}$ denotes the L_∞ space (pt, \mathfrak{g}) ; sometimes we will also use the notation for the dg-manifold $(pt, C^*(\mathfrak{g}))$.

⁵Obviously there are various hypotheses (such as finiteness conditions) that need to be satisfied to apply this process, but this proposition applies in many situations.

as a $C^\#(\mathfrak{g})$ -module. This trivialization equips \mathcal{M} with a connection

$$\begin{aligned} C^\#(\mathfrak{g}) \otimes_k M &\rightarrow \Omega_{B\mathfrak{g}}^1 \otimes_k M, \\ f \otimes m &\mapsto (d_{dR}f) \otimes m. \end{aligned}$$

Define $\text{At}(\mathcal{M})$ to be the Atiyah class for this connection.

The Atiyah class lives in $\Omega_{B\mathfrak{g}}^1(\text{End } \mathcal{M}) \cong C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1] \otimes \text{End}_k(M))$. We can thus view it as a map

$$\text{At}(\mathcal{M}) : T_{B\mathfrak{g}} \otimes \mathcal{M} \rightarrow \mathcal{M}$$

and ask for the Taylor coefficients as a section of $B\mathfrak{g}$.

Proposition 5.4 (Proposition 6.4 of [GGb]). *Given $x \in \mathfrak{g}$, we obtain a vector field X on $B\mathfrak{g}$, by shifting the degree of x . Let m be a section in \mathcal{M} . We find*

$$\text{At}(\mathcal{M})(X \otimes m) = \ell_2(x, m) + \ell_3(x, x, m) + \cdots + \ell_n(x^{\otimes n-1}, m) + \cdots.$$

Alternatively, we say that for X a vector field, $m \in \mathcal{M}$, and $x_1, \dots, x_n, y \in \mathfrak{g}$,

$$\left. \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \right|_0 \text{At}(\mathcal{M})(X \otimes m) = \ell_{n+2}(x_1, \dots, x_n, x, m) \in \mathfrak{g},$$

where $x \in \mathfrak{g}$ is the shift of x .

5.3 Useful facts about the Atiyah class

Let us recall a few facts about the Atiyah class. See Section 6.3 of [GGb] for the corresponding proofs.

Proposition 5.5. *The Atiyah class is closed: $d_{\Omega^1 \otimes \text{End } M} \text{At}(\nabla) = 0$.*

In analogy with geometry, the de Rham differential d_{dR} extends to a complex Ω_R^* , with exterior derivative $d_{dR} : \Omega_R^k \rightarrow \Omega_R^{k+1}$ such that $[d, d_{dR}] = 0$, where d denotes the differential on the R -modules Ω_R^k .

We are now led to the following question: what if ∇ equips M with a flat connection, so that $\nabla^2 = 0$? In that case, ∇ makes $\Omega_R^* \otimes_R M$ a cochain complex over $R^\#$, the underlying graded algebra. Hence the Atiyah class is the obstruction to making M a “vector bundle with flat connection” over the space described by R . This situation is precisely what appears in our jet-bundle approach to the Chern-Weil construction of characteristic classes which we recall in the next section. The Atiyah class will play the same role that the curvature usually does because it will be precisely the obstruction to making the connection flat.

In this situation, we have a natural analogue of the Bianchi identity. Recall that a connection ∇ on M induces a connection ∇^{End} on $\text{End } M$.

Proposition 5.6. *If $\nabla^2 = 0$, then $\text{At}(\nabla)$ is a horizontal section of $\Omega_R^* \otimes_R \text{End } M$. More explicitly,*

$$\nabla^{\text{End}} \text{At}(\nabla) = 0.$$

6 Characteristic classes

6.1 The Chern character

In representation theory, the character of a representation is one of the most useful invariants; in geometry, the Chern character of a bundle is likewise one of the most useful invariants.⁶ In this section and the next, we want hybridize these constructions to define a Chern character and certain characteristic classes in the setting of dg-manifolds/ L_∞ spaces.

Definition 6.1. The *Chern character* of a connection ∇ is $ch(\nabla) := \text{Tr} \exp \left(\frac{\text{At}(\nabla)}{-2\pi i} \right)$.

We let $ch_k(\nabla)$ denote the homogeneous component of $ch(\nabla)$ in Ω_R^k . Hence $ch_k(\nabla) = \text{Tr} \left(\frac{1}{k!(-2\pi i)^k} \text{At}(\nabla)^k \right)$.

As stated, the Chern character is an element in Ω_R^* of mixed degree, but it is more natural (as we explain below) to make it homogeneous by forcing $\text{At}(\nabla)$ to be homogeneous as follows. Observe that $\text{At}(\nabla)$ lives in $\Omega_R^1 \otimes_R \text{End } M$, and it has degree 1. We can identify it with a degree 0 element if we instead view it as living in $\Omega_R^1 \otimes_R \text{End } M[1]$. In that case, the powers $\text{At}(\nabla)^k$ live in $\Omega_R^k \otimes_R \text{End } M[k]$ and have degree 0. The Chern character $ch(\nabla)$ is then a homogeneous, degree 0 element of $\oplus_k \Omega_R^k[k]$, which we will denote as Ω_R^{-*} . From the perspective of derived geometry, this setting is more natural since we only access homogeneous elements when we work functorially (cf. Bernstein's discussion of the "even rules" principle in [DEF⁺99]).

Corollary 6.2. If $\nabla^2 = 0$, then the Chern classes $ch_k(\nabla)$ are closed under d_{dR} and d_{Ω^k} .

Proof. Both of these follow straightforwardly from our work in the preceding section. □

6.2 Genera: $\hat{A}(Bg)$

Recall that (following Hirzebruch [Hir95]) the Todd class can be defined in terms of Chern classes by the power series $Q(x)$ and the \hat{A} class is given in Pontryagin classes via $P(x)$ where

$$Q(x) = \frac{x}{1 - e^{-x}} \quad \text{and} \quad P(x) = \frac{x/2}{\sinh x/2}.$$

We define a new power series by $\log(Q(x)) - x/2$ and denote the corresponding characteristic class by $\log(e^{-c_1/2} \text{Td})$. We have an equivalence of power series (see [WML92] and [HB92])

$$\log \left(\frac{x}{1 - e^{-x}} \right) - \frac{x}{2} = \sum_{k \geq 1} 2\zeta(2k) \frac{x^{2k}}{2k(2\pi i)^{2k}}, \quad (6.1)$$

where ζ is the Riemann zeta function.

We now use standard arguments about characteristic classes. For a sum of complex line bundles $E = L_1 \oplus \cdots \oplus L_n$, the Todd class is

$$\text{Td}(E) = Q(c_1(L_1)) \cdots Q(c_1(L_n)).$$

⁶For a further discussion of the relationship between representation theory and the Chern character, see for instance [TV09].

Thus, equation 6.1 tells us

$$\log(e^{-c_1(E)/2} Td(E)) = \sum_{k \geq 1} \frac{2\zeta(2k)}{2k(2\pi i)^{2k}} (c_1(L_1)^{2k} + \cdots + c_1(L_n)^{2k}).$$

As $ch_{2k}(E) = (c_1(L_1)^{2k} + \cdots + c_1(L_n)^{2k}) / (2k!)$, we obtain a general formula for an arbitrary bundle E ,

$$\log(e^{-c_1(E)/2} Td(E)) = \sum_{k \geq 1} \frac{2\zeta(2k)}{2k(2\pi i)^{2k}} (2k)! ch_{2k}(E).$$

Putting together the above discussion we make the following definition.

Definition 6.3. Let V be a vector bundle over (X, \mathfrak{g}) (e.g. the tangent bundle as given by the module $\mathfrak{g}[1]$) then we define

$$\log(\hat{A}(V)) \stackrel{def}{=} \sum_{k \geq 1} \frac{2\zeta(2k)}{2k(2\pi i)^{2k}} (2k)! ch_{2k}(V) \in \Omega_{B\mathfrak{g}}^{-*}.$$

7 L_∞ spaces from smooth manifolds

7.1 Encoding the smooth geometry of a manifold as an L_∞ space

Given a smooth manifold X we can form the naive dg-manifold (X, C_X^∞) , but we are more interested in dg-manifolds which live over $X_\Omega = (X, \Omega_X)$ which allows us to use formal geometric constructions of characteristic classes in the language of Atiyah classes (as in the previous section). There is a nice replacement for X as a dg-manifold which uses the infinite jet bundle of functions on X . Further, this dg-manifold is obtained from an L_∞ space built out of the shifted tangent bundle of X , $T_X[-1]$.

The following lemma makes the picture above precise.

Lemma 7.1 (Lemma 9.1 of [GGb]). *There is a curved L_∞ algebra \mathfrak{g}_X over Ω_X , with nilpotent ideal $\Omega_X^{>0}$, such that*

1. $\mathfrak{g}_X \cong T_X[-1] \otimes_{C_X^\infty} \Omega_X^\#$ as an $\Omega_X^\#$ module;
2. $C^*(\mathfrak{g}_X) \cong dR(\mathcal{J})$ as commutative Ω_X algebras;
3. $C^*(\mathfrak{g}_X) \simeq C_X^\infty$ as Ω_X modules.

The L_∞ algebra \mathfrak{g}_X is well defined up to equivalence, see Proposition 17.4 of [Gra].

Definition 7.2. Let (X, \mathfrak{g}_X) be the L_∞ space encoding the smooth geometry of X , then we define the dg-manifold

$$B\mathfrak{g}_X \stackrel{def}{=} (X, C^*(\mathfrak{g}_X))$$

7.1.1 The S^1 action

In the sequel we will consider the L_∞ algebra

$$\Omega_{S^1}^* \otimes (\mathfrak{g}_X \oplus \mathfrak{g}_X^\vee[-2]) \cong \mathbb{C}[\epsilon] \otimes (\mathfrak{g}_X \oplus \mathfrak{g}_X^\vee[-2])$$

where ϵ is a square zero parameter of degree 1. We have seen in Section 4.2 that this L_∞ algebra is the structure sheaf of the derived loop space

$$\mathcal{L}T^*(X, \mathfrak{g}_X) \cong T[-1]T^*(X, \mathfrak{g}_X).$$

Now by definition functions $\mathcal{O}(T[-1]T^*(X, \mathfrak{g}_X)) = \Omega_{T^*B\mathfrak{g}}^{-*}$, that is $\Omega_{T^*B\mathfrak{g}}^k$ is in degree $-k$. There is no de Rham differential, just the internal differential coming from the Ω_X^* -algebra structure. In order to restore the de Rham we need to notice the $\mathbb{C}[\epsilon]$ action $\mathcal{L}T^*(X, \mathfrak{g}_X)$, we actually prefer to think of this as an action of the dg-manifold $B\mathbb{G}_a = (pt, \mathbb{C}[\epsilon])$. The de Rham differential corresponds to the L_∞ algebra derivation $\partial/\partial\epsilon$.

These constructions are well-known, usually referred to by the name of *mixed complexes* or *cyclic modules* (see [BZN] and [TV09]). If we ask for the $B\mathbb{G}_a$ -invariant functions on $\mathcal{L}T^*(X, \mathfrak{g}_X)$, we obtain the *negative cyclic homology* of $T^*(X, \mathfrak{g}_X)$. For a thorough discussion of these ideas in the language of derived geometry, see [BZN] and [TV09]. We emphasize these circle actions here as they are crucial for actually recovering the \hat{A} -class in smooth geometry.

7.2 The L_∞ structure associated to a vector bundle

Let E be a vector bundle over X whose sections we denote \mathcal{E} . In this section we will set up the algebraic preliminaries to perform a Chern-Weil style construction of characteristic classes in the next section. The main output of this section will be the encoding of a vector bundle

Recall that infinite jet bundle $\mathcal{J}(E)$ has stalk at a point $x \in X$

$$\mathcal{J}(E)_x = \varprojlim \mathcal{E} / \mathfrak{m}_x^k \mathcal{E},$$

where \mathfrak{m}_x denotes the maximal ideal of functions vanishing at x . Note that this equips $\mathcal{J}(E)$ with a \mathcal{J} -module (in particular a C_X^∞ -module) structure as well as a filtration

$$\mathcal{J}(E) = F^0 \mathcal{J}(E) \supset F^1 \mathcal{J}(E) \supset F^2 \mathcal{J}(E) \supset \dots$$

by the order of vanishing. We have that $F^0/F^1 \cong \mathcal{E}$ canonically, let $\text{Split}(\mathcal{J}(E))$ denote the space of C_X^∞ splittings, i.e. $\sigma : \mathcal{E} \rightarrow \mathcal{J}(E)$ so that

$$0 \longrightarrow F^1 \mathcal{J}(E) \longrightarrow \mathcal{J}(E) \xrightarrow{\sigma} \mathcal{E} \longrightarrow 0$$

Note that we have the map of sheaves of \mathbb{R} -algebras $J : \mathcal{E} \rightarrow \mathcal{J}(E)$, but this map is not C_X^∞ -linear as the taylor series of a product is the product of the taylor series (one can also see the failure of J to be a vector bundle map by looking locally).

Let \mathfrak{J} denote the de Rham complex jets and $d_{dR} : \mathfrak{J} \rightarrow \Omega_{\mathfrak{J}}^1$ its universal derivation. If we define

$$\Omega_{\mathfrak{J}}^p = \bigwedge^p \Omega_{\mathfrak{J}}^1$$

where the exterior algebra is over Ω_X , then we have a complex of Ω_X -modules

$$\mathfrak{J} \xrightarrow{d_{dR}} \Omega_{\mathfrak{J}}^1 \xrightarrow{d_{dR}} \Omega_{\mathfrak{J}}^2 \xrightarrow{d_{dR}} \dots \xrightarrow{d_{dR}} \Omega_{\mathfrak{J}}^{\dim X}.$$

We now apply Lemma D.1 to this complex to obtain the following.

Lemma 7.3. *As Ω_X -modules, $\Omega_{\mathfrak{J}}^k \cong dR(J(\Omega_X^k))$. Consequently, $H^0(\Omega_{\mathfrak{J}}^k) \cong \Omega_X^k$ and $J : \Omega_X^* \hookrightarrow \Omega_{\mathfrak{J}}^*$ is a quasi-isomorphism of complexes.*

Now Let $\mathfrak{J}(E)$ denote the de Rham complex of the D -module $\mathscr{J}(E)$. From general non sense about jets, specifically Lemma C.1, we have the following.

Lemma 7.4. *Let $\sigma \in \text{Split}(\mathscr{J}(E))$, then σ induces a C_X^∞ -linear isomorphism*

$$i_\sigma : \mathcal{E} \otimes_{C_X^\infty} \mathscr{J} \xrightarrow{\sim} \mathscr{J}(E), \quad i_\sigma(s \otimes j) = j \cdot \sigma(s).$$

This isomorphism extends to the associated de Rham complexes

$$i_\sigma : \mathcal{E} \otimes_{C_X^\infty} \mathfrak{J} \xrightarrow{\sim} \mathfrak{J}(E).$$

Lemma 7.5. *Let $\sigma \in \text{Split}(\mathscr{J}(E))$, then σ induces a flat connection ∇_σ on $\mathfrak{J}(E)$.*

Proof. In order to define the connection $\nabla_\sigma : \mathfrak{J}(E) \rightarrow \mathfrak{J}(E) \otimes_{\mathfrak{J}} \Omega_{\mathfrak{J}}^1$ we use the isomorphism $i_\sigma : \mathfrak{J}(E) \cong \mathcal{E} \otimes_{C_X^\infty} \mathfrak{J}$ and the resulting isomorphism $\mathfrak{J}(E) \otimes_{\mathfrak{J}} \Omega_{\mathfrak{J}}^1 \cong \mathcal{E} \otimes_{C_X^\infty} \Omega_{\mathfrak{J}}^1$. Indeed,

$$\nabla_\sigma : \mathcal{E} \otimes_{C_X^\infty} \mathfrak{J} \rightarrow \mathcal{E} \otimes_{C_X^\infty} \Omega_{\mathfrak{J}}^1, \quad s \otimes j \mapsto s \otimes d_{dR}(j)$$

defines a connection. That $(\nabla_\sigma)^2 = 0$ follows from the fact that d_{dR} is square zero. \square

We now have a cochain complex of Ω_X^\sharp modules

$$\mathfrak{J}(E) \xrightarrow{\nabla_\sigma} \mathfrak{J}(E) \otimes_{\mathfrak{J}} \Omega_{\mathfrak{J}}^1 \xrightarrow{\nabla_\sigma} \mathfrak{J}(E) \otimes_{\mathfrak{J}} \Omega_{\mathfrak{J}}^2 \rightarrow \dots$$

There is no reason for the this connection to be compatible with the Ω_X module structure on these sheaves. In fact the failure of this compatibility is measured by the Atiyah class of our connection $\text{At}(\nabla_\sigma)$. Note that

$$\text{At}(\nabla_\sigma) \in \Omega_{\mathfrak{J}}^1 \otimes_{\mathfrak{J}} \text{End}_{\mathfrak{J}}(\mathfrak{J}(E))$$

and is of cohomological degree 1, and so we have the k th power of the Atiyah class

$$\text{At}(\nabla_\sigma)^k \in \Omega_X^k \left(\Omega_{\mathfrak{J}}^k \otimes_{\mathfrak{J}} \text{End}_{\mathfrak{J}}(\mathfrak{J}(E)) \right).$$

Let us define

$$\omega_k \stackrel{\text{def}}{=} \text{Tr} \left(\text{At}(\nabla_\sigma)^k \right) \in \Omega_X^k(\Omega_{\mathfrak{J}}^k).$$

We will use these classes ω_k in the next section define characteristic classes of the vector bundle E . The following proposition shows that they can used in the Chern-Weil construction of characteristic classes.

Proposition 7.6 (Proposition 11.1 of [GGb]). *The Chern class $ch_k(E) \in H^{2k}(X)$ is given by $\frac{1}{k!(-2\pi i)^k} \text{Tr} \left((\text{At}(\nabla_\sigma))^k \right)$*

7.3 The characteristic class $\hat{A}_u(X)$

In encoding X as $B\mathfrak{g}_X$, we use the formalism of Gelfand-Kazhdan formal geometry to construct \mathfrak{g}_X ; essentially, we replace smooth functions C_X^∞ by the de Rham complex of jets of smooth functions. As a result, our construction of the global observables involves a complex quasi-isomorphic to (shifted) de Rham forms, and the characteristic classes $ch_k(B\mathfrak{g}_X)$ all manifestly have cohomological degree 0 in this construction. Thus the difficulty is in identifying $ch_k(B\mathfrak{g}_X)$ with $ch_k(X)$, and the negative cyclic homology achieves this.

Just as we saw with $\Omega_{T^*B\mathfrak{g}_X}^{-*}$, $\Omega_{B\mathfrak{g}_X}^{-*}$ is a $B\mathbb{G}_a$ dg-module (here we have no internal differential and view d_{dR} as lowering degree by 1). Now by the discussion in Section 7.1.1 we have a quasi-isomorphism of complexes of Ω_X^* -modules

$$(\Omega_{B\mathfrak{g}_X}^{-*})^{B\mathbb{G}_a} \simeq dR(J(\Omega_X^{-*}[[u]], ud)).$$

Recall the characteristic classes $ch_{2k}(\nabla_\sigma) \in \Omega_X^{2k}(\Omega_{\mathfrak{J}}^{2k})$. This double complex has acyclic columns, so we want to zig-zag to a class on the bottom row. That is, $ch_{2k}(\nabla_\sigma)$ is closed with respect to both the horizontal differential (which in this case is the de Rham differential) and the vertical differential (the one coming from the jet bundle) and hence its cohomology class in the total complex is represented by a class of α_1 of cohomological degree $2k - 1$ in $\Omega_{\mathfrak{J}}^{2k+1}$. Continuing in this manner we obtain a class $\alpha_{2k} \in H^0(\Omega_{\mathfrak{J}}^{4k}) \cong \Omega_X^{4k}$.

Now we want to identify the image of the class α_{2k} in the complex $(\Omega_{B\mathfrak{g}_X}^{-*})^{B\mathbb{G}_a}$. From Lemma 7.3 we have that $\Omega_{\mathfrak{J}}^k \cong dR(J(\Omega_X^k))$ as Ω_X^* -modules. Let $\tilde{ch}_{2k}(\nabla_\sigma)$ denote the class

$$\frac{1}{(2k)!(-2\pi i)^{2k}} \text{Tr}(\text{At}(\nabla_\sigma)^{2k}) \in \Omega_X^{2k}(\Omega_{\mathfrak{J}}^{2k})[2k] \subset \Omega_{B\mathfrak{g}_X}^{-*}.$$

In order to enact the zig-zag argument (and hence produce a nontrivial cohomology class) we need the de Rham differential which is obtained on $\Omega_{B\mathfrak{g}_X}^{-*}$ by taking (homotopy) invariants with respect to the action of $B\mathbb{G}_a$. As we zig-zag down to row 0 (that is $\Omega_X^0(\Omega_{\mathfrak{J}}^{-*})$) we pick up a factor of u at each step. Therefore, if we denote resulting class by $\tilde{\alpha}_{2k}$ we have that

$$\tilde{ch}_{2k}(\nabla_\sigma) \simeq \tilde{\alpha}_{2k} \simeq u^{2k} \alpha_{2k} \simeq u^{2k} ch_{2k}(X) \in \Omega_X^0(\Omega_{\mathfrak{J}}^{4k})[[u]][4k] \subset (\Omega_{B\mathfrak{g}_X}^{-*})^{B\mathbb{G}_a}.$$

Following the presentation of Section 6.2 we define for any smooth manifold X the class $\log(\hat{A}_u(X))$ to be

$$\log(\hat{A}_u(X)) \stackrel{def}{=} \sum_{k \geq 1} \frac{2(2k-1)!}{(2\pi i)^{2k}} u^{2k} \zeta(2k) ch_{2k}(X) \in (\Omega_X^{-*}[[u]], ud).$$

This is the usual logarithm of the \hat{A} class weighted by powers of u . So far we have argued that $\tilde{ch}_{2k}(\nabla_\sigma) \simeq u^{2k} ch_{2k}(X) \in dR(J(\Omega_X^{-*}[[u]], ud))$. Further, it is shown in Section 20.2 of [Gra] that

$$dR(J(\Omega_X^{-*}[[u]], ud)) \simeq (\Omega_X^{-*}[[u]], ud).$$

8 One-dimensional Chern-Simons theory and its quantization

Here we briefly describe a family of field theories which we call one-dimensional Chern-Simons theories. We are working in Costello's paradigm of effective BV theory as described in [Cos11], see also [GGB] for further details on Chern-Simons type theories.

Although Chern-Simons theory typically refers to a gauge theory on a 3-manifold, the perturbative theory has analogues over a manifold of any dimension. The only modification is to use dg Lie algebras, or L_∞ algebras, with an invariant pairing of the appropriate degree.

8.1 The simplest example

Our base space is S^1 . Let $\mathfrak{h} = \bigoplus_n \mathfrak{h}_n$ be a graded Lie algebra with a nondegenerate invariant symmetric pairing $\langle -, - \rangle_{\mathfrak{h}}$ of degree -2. Notice that this means $\mathfrak{h}[1]$ comes equipped with a nondegenerate skew-symmetric pairing of degree 0, which we denote $\langle -, - \rangle_{\mathfrak{h}[1]}$. The space of fields is $\mathcal{E} = \Omega_{S^1}^* \otimes \mathfrak{h}[1]$. The pairing on \mathfrak{h} induces a symplectic form of degree -1 on \mathcal{E} by

$$\langle \alpha, \beta \rangle = \int_{t \in S^1} \langle \alpha(t) \wedge \beta(t) \rangle_{\mathfrak{h}[1]}.$$

More explicitly we choose a metric on S^1 and let $\alpha = \sum_n A_n^0(t) + A_n^1(t)dt$ denote an element of \mathcal{E} , where $A_n^0(t)$ and $A_n^1(t)$ are smooth functions on S^1 taking values in $\mathfrak{h}[1]_n$, and likewise for $\beta = \sum B_n^0(t) + B_n^1(t)dt$. Then

$$\langle \alpha, \beta \rangle = \sum_n \int_{t \in S^1} \langle A_n^0(t), B_{-n}^1(t) \rangle_{\mathfrak{h}[1]} + \langle A_n^1(t), B_{-n}^0(t) \rangle_{\mathfrak{h}[1]} dt.$$

Now let $Q = d$, the exterior derivative, and $Q^* = d^*$, its adjoint with respect to our metric. The action functional is

$$S(\alpha) = \frac{1}{2} \langle \alpha, d\alpha \rangle + \frac{1}{6} \langle \alpha, [\alpha, \alpha] \rangle.$$

8.2 The general case

Let \mathfrak{g} now denote a curved L_∞ algebra over a commutative dga R . Let the maps $\ell_n : \wedge^n \mathfrak{g} \rightarrow \mathfrak{g}$ denote the brackets (i.e., these are the Taylor components of the derivation $d_{\mathfrak{g}}$ defining the L_∞ structure). We want an L_∞ algebra that has a nondegenerate invariant symmetric pairing $\langle -, - \rangle$ of degree -2. Note that the sum $\mathfrak{g} \oplus \mathfrak{g}^\vee[-2]$ is equipped with an L_∞ structure using the coadjoint action:

$$[X + \lambda, Y + \mu] = [X, Y] + X \cdot \mu - Y \cdot \lambda,$$

where $X, Y \in \mathfrak{g}$ and $\lambda, \mu \in \mathfrak{g}^\vee[-2]$. Moreover, $\mathfrak{g} \oplus \mathfrak{g}^\vee[-2]$ also has a natural pairing

$$\langle X + \lambda, Y + \mu \rangle = \lambda(Y) - \mu(X),$$

which is invariant by construction.

Our space of fields is

$$\Omega_{S^1} \otimes (\mathfrak{g}[1] \oplus \mathfrak{g}^\vee[-1]).$$

Our action functional is

$$S(\phi) = \frac{1}{2} \langle \phi, d\phi \rangle + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \langle \phi, \ell_n(\phi^{\otimes n}) \rangle.$$

Note that when \mathfrak{g} is just a graded Lie algebra, ℓ_2 is the only nontrivial bracket and we recover the action functional from the simple example above. The Euler-Lagrange equation of S is the Maurer-Cartan equation for the trivial $\mathfrak{g} \oplus \mathfrak{g}^{\vee}[-2]$ -bundle on S^1 .

Remark 8.1. We view the action S as a sum of a free action functional $\frac{1}{2} \langle \phi, (d + l_1)\phi \rangle$ and an interaction term $I_{CS} = \sum_{n \neq 1} \frac{1}{(n+1)!} \langle \phi, \ell_n(\phi^{\otimes n}) \rangle$.

8.3 Quantization

Here we recall the operadic approach to quantization as given in [CG].

Definition 8.2. A P_0 algebra is a cdga P with a poisson bracket of degree $+1$.

Lemma 8.3. $\mathcal{O}(T^*[-1](X, \mathfrak{g}))$ is a P_0 algebra.

Proof. We have

$$\mathcal{O}(T^*[-1](X, \mathfrak{g})) = \widehat{\text{Sym}}((\mathfrak{g} \oplus \mathfrak{g}^{\vee}[-3])^{\vee}[-1]) = \widehat{\text{Sym}}(\mathfrak{g}^{\vee}[-1] \oplus \mathfrak{g}[2]).$$

Now $\mathfrak{g} \oplus \mathfrak{g}^{\vee}[-3]$ has a pairing $\langle -, - \rangle$ of degree -3 and the poisson bracket $\{ -, - \}$ on $\mathcal{O}(T^*[-1](X, \mathfrak{g}))$ is generated by $\langle -, - \rangle$ now viewed as a degree $+1$ pairing. \square

Similarly, the global classical observables of a BV theory in the sense of [Cos11] form a P_0 algebra with bracket induced by the degree -1 symplectic pairing on fields.

Definition 8.4.

1. A *BD algebra* is a cochain complex (A, d) flat over $\mathbb{C}[[\hbar]]$ with commutative product \star and poisson bracket $\{ -, - \}$ of degree $+1$ such that

$$d(a \star b) = (da) \star b \pm a \star (db) + \hbar \{a, b\}.$$

2. A *quantization* of a P_0 algebra P is a BD algebra A such that A modulo \hbar is P .⁷

It is true that the global quantum observables of a BV theory form a BD algebra. Further, it is true that the global quantum observables are a quantization of the global classical observables.

Theorem 8.5 (Global version of main theorem of [CG]). *Given a BV theory $(\mathcal{E}, \langle -, - \rangle_{loc}, Q, Q^*, \{I[L]\})$, then Obs^q is a quantization of the P_0 algebra Obs^{cl} .*

⁷If the P_0 algebra is equipped with a \mathbb{C}^{\times} action where the bracket has weight $+1$ we can ask for \mathbb{C}^{\times} -equivariant quantization which means that \hbar has weight -1 . We can equip $T^*[-1](X, \mathfrak{g})$ with an action of \mathbb{C}^{\times} by acting on the factor $\mathfrak{g}^{\vee}[-3]$ which we think of as scaling the cotangent fiber.

8.4 Quantization of one-dimensional Chern-Simons

We recall the main theorems from [GGb] which describe the global classical and quantum observables for one-dimensional Chern-Simons theory determined by an L_∞ algebra \mathfrak{g} . (It is non-trivial to see that a quantization exists after which we can begin to identify the complex of global observables.) To begin, we note that the classical observables on S^1 are given by the commutative dg algebra

$$\left(\widehat{\text{Sym}} \left((\Omega_{S^1}^* \otimes (\mathfrak{g}[1] \oplus \mathfrak{g}^\vee[-1]))^\vee \right), d + \{I_{CS}, -\} \right),$$

where I_{CS} is the interacting part of the action functional as described in section 8.2.

Theorem 8.6. *The global quantum observables of the Chern-Simons theory determined by \mathfrak{g} on S^1 are quasi-isomorphic to the following cochain complex:*

$$(\Omega_{T^*B\mathfrak{g}}^{-*}[[\hbar]], \hbar L_\Pi + \hbar \{\log(\hat{A}(B\mathfrak{g})), -\}),$$

where L_Π denotes the Lie derivative with respect to the canonical Poisson bivector Π on $T^*B\mathfrak{g}$.

Corollary 8.7. *Let X be a Complex manifold and $\mathfrak{g}_{X_{\bar{\partial}}}$ the L_∞ algebra which encodes the complex geometry of X . The global quantum observables on S^1 for the one-dimensional Chern-Simons theory determined by $\mathfrak{g}_{X_{\bar{\partial}}}$ are quasi-isomorphic to*

$$\left(\Omega_{hol}^{-*}(T^*X)[[\hbar]], \hbar L_\Pi + \hbar \left\{ \log \left(e^{-c_1(X)/2} \text{Td}(X) \right), - \right\} \right),$$

where Ω_{hol}^k denotes the holomorphic k -forms.

When our L_∞ algebra encodes the smooth geometry of a manifold X it is necessary to take into account the homotopy S^1 action.

Theorem 8.8. *Consider the one-dimensional Chern-Simons theory determined by \mathfrak{g}_X , where \mathfrak{g}_X is the L_∞ algebra which encodes the smooth geometry of X . The S^1 -invariant global quantum observables over S^1 form a cochain complex quasi-isomorphic to the following deformation of the negative cyclic homology of T^*X :*

$$(\Omega_{T^*X}^{-*}[[u]][[\hbar]], u d + \hbar L_\Pi + \hbar \{\log(\hat{A}_u(X)), -\}).$$

9 Projective volume forms

In this section we discuss the notion of projective volume forms on L_∞ spaces and report on our progress in interpreting $\hat{A}(B\mathfrak{g})$ as such a volume form on $T[-1](X, \mathfrak{g})$.

Let (X, \mathfrak{g}) be an L_∞ space and $B\mathfrak{g}$ the associated dg manifold. Motivated by complex geometry, Costello [Cosb] makes the following definition.

Definition 9.1. A *projective volume form* on (X, \mathfrak{g}) (equivalently on $B\mathfrak{g}$) is a right $D(B\mathfrak{g})$ -module structure on $\mathcal{O}(B\mathfrak{g})$.

Here $D(B\mathfrak{g})$ is the associative algebra of differential operators on $B\mathfrak{g}$. Recall that the dg Lie algebra of vector fields on $B\mathfrak{g}$ is given by (see Section 4)

$$\mathrm{Vect}(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}[1]) = \mathrm{Der}(\mathcal{O}(B\mathfrak{g})).$$

Then we define $D(B\mathfrak{g})$ to be the free associative algebra over $\mathcal{O}(B\mathfrak{g})$ generated by $\chi \in \mathrm{Vect}(B\mathfrak{g})$ subject to the relations:

$$\begin{aligned}\chi \cdot f - f \cdot \chi &= (\chi f) \\ f \cdot \chi &= f\chi\end{aligned}$$

where \cdot denotes the associative product in $D(B\mathfrak{g})$ and juxtaposition indicates the action of $\mathrm{Vect}(B\mathfrak{g})$ on $\mathcal{O}(B\mathfrak{g})$ by derivations or the module structure of $\mathrm{Vect}(B\mathfrak{g})$ over $\mathcal{O}(B\mathfrak{g})$.

Proposition 9.2 (Proposition 11.7.1 of [Cosb]). *There is a bijection between the set of right $D(B\mathfrak{g})$ -structures on $\mathcal{O}(B\mathfrak{g})$ and that of \mathbb{C}^\times -equivariant quantizations of the P_0 algebra $\mathcal{O}(T^*[-1]B\mathfrak{g})$.*

The rough idea of this proposition is illustrated by the following smooth manifold example. Let X be a smooth n -manifold with volume form $d\mathrm{Vol}$, then we have following commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega_X^{n-2} & \xrightarrow{d} & \Omega_X^{n-1} & \xrightarrow{d} & \Omega_X^n \xrightarrow{f} \mathbb{C} \\ & & \uparrow i_{d\mathrm{Vol}} & & \uparrow i_{d\mathrm{Vol}} & & \uparrow i_{d\mathrm{Vol}} \\ \cdots & \longrightarrow & \Lambda^2 T_X[2] & \xrightarrow{\Delta} & T_X[1] & \xrightarrow{\Delta} & C_X^\infty \end{array}$$

The significance of the diagram is that $\Delta = \mathrm{div}_{d\mathrm{Vol}}$ and the bottom row is functions on $T^*[-1]X$ i.e. poly vector fields. Poly vector fields have the P_0 structure given by the Schouten bracket and the bottom row is a quantization of this structure (after tensoring with $\mathbb{C}[[\hbar]]$) where the new bracket is given by

$$\{a, b\} = \Delta(\alpha\beta) - (\Delta\alpha)\beta - (-1)^{|\alpha|}\alpha(\Delta\beta).$$

For pedagogical reasons, we record Costello's proof of the proposition.

Proof of Proposition 9.2. Suppose we have a right $D(B\mathfrak{g})$ -module structure on $\mathcal{O}(B\mathfrak{g})$. If $V \in D(B\mathfrak{g})$ and $f \in \mathcal{O}(B\mathfrak{g})$, we let $f\rho(V) \in \mathcal{O}(B\mathfrak{g})$ be the result of applying V to f using the right D -module structure. Note that, by definition, for $g \in \mathcal{O}(B\mathfrak{g}) \subset D(B\mathfrak{g})$, $g\rho(f) = gf$. Thus, for $\chi \in \mathrm{Vect}(B\mathfrak{g})$,

$$f\rho(\chi) = 1\rho(f)\rho(\chi) = 1\rho(f\chi).$$

Thus, the entire action is determined by a linear map

$$\begin{aligned}\Phi : \mathrm{Vect}(B\mathfrak{g}) &\rightarrow \mathcal{O}(B\mathfrak{g}) \\ \Phi(\chi) &= 1\rho(\chi)\end{aligned}$$

Note that the relations in $D(B\mathfrak{g})$ imply that $\chi \cdot f = (\chi f) + (-1)^{|f||\chi|}f\chi$. It follows that

$$\Phi(f\chi) - f\Phi(\chi) = -(-1)^{|f||\chi|}(\chi f) \in \mathcal{O}(B\mathfrak{g}). \quad (\dagger)$$

We use the map Φ to define a quantization of $\mathcal{O}(T^*[-1]B\mathfrak{g})$. The underlying graded Poisson algebra of our quantization is $\mathcal{O}(T^*[-1]B\mathfrak{g})[[\hbar]]$. To describe the differential, let us introduce an auxiliary operator Δ on $\mathcal{O}(T^*[-1]B\mathfrak{g})$. The operator Δ is the unique order two differential operator with the property that, for $f \in \mathcal{O}(B\mathfrak{g})$, and for $\chi \in \text{Vect}(B\mathfrak{g})[1] \subset \mathcal{O}(T^*[-1]B\mathfrak{g})$, we have

$$\begin{aligned}\Delta(f) &= 0 \\ \Delta(\chi) &= \Phi(\chi).\end{aligned}$$

The fact that Δ is well-defined follows from the fact that Φ is an order one differential operator. It is not hard to verify (from (+)) that the failure of Δ to be a derivation is the Poisson bracket on $\mathcal{O}(T^*[-1]B\mathfrak{g})$. Thus, we define the differential on our BD algebra to be $d + \hbar\Delta$, where d is the usual differential on $\mathcal{O}(T^*[-1]B\mathfrak{g})$.

Let us now consider the converse. That our quantization is \mathbb{C}^\times invariant forces the differential to be of the form $d + \hbar\Delta$, where d is the given differential on $\mathcal{O}(T^*[-1]B\mathfrak{g})$, that is the differential on the underlying P_0 algebra. Δ is then some operator which maps

$$\Gamma(B\mathfrak{g}, \Lambda^i TB\mathfrak{g}) \rightarrow \Gamma(B\mathfrak{g}, \Lambda^{i-1} TB\mathfrak{g}).$$

The operator Δ is determined uniquely by its behavior on $\text{Vect}(B\mathfrak{g})$; restricted to this subspace, it must be a cochain map

$$\Phi_\Delta : \text{Vect}(B\mathfrak{g}) \rightarrow \mathcal{O}(B\mathfrak{g})$$

satisfying the axiom in (+). □

Remark 9.3.

1. The right $D(B\mathfrak{g})$ -module structures on $\mathcal{O}(B\mathfrak{g})$ actually form a simplicial set if we consider families of such objects parameterized by $\Omega^*(\Delta^n)$. Similarly, we can define quantization in families which are linear over $\Omega^*(\Delta^n)$. In this setting, Proposition 9.2 is a homotopy equivalence of simplicial sets. We have chosen to present it as a bijection after applying the functor π_0 .
2. Projective volume forms continue to make sense in the setting where the Lie algebroid of vector fields is replaced by an arbitrary Lie algebroid. This perspective is explained in [CG], but is unnecessary for our current aims.

Continuing with the smooth manifold example, we have a map of sheaves of complexes

$$C_X^\infty \rightarrow (\mathcal{O}(T^*[-1]X), \Delta) \cong \mathbb{C}[n]$$

where the last isomorphism is only determined up to a scalar. Passing to compactly supported sections we obtain the standard integration

$$\int : C_X^\infty \rightarrow H_c^n(X, \mathbb{C}) = \mathbb{C}.$$

As we show below, the same holds for *nice* L_∞ spaces (X, \mathfrak{g}) , that is we have a map

$$\int : \mathcal{O}(X, \mathfrak{g}) \rightarrow (\mathcal{O}(T^*[-1](X, \mathfrak{g}))((\hbar)), d + \hbar\Delta).$$

10 Integrability

We will be mostly interested in restricting to L_∞ spaces that are not too wild. Following [Cosa] we make the following definitions.

Definition 10.1.

1. An L_∞ space (X, \mathfrak{g}) is *locally trivial* if the C_X^∞ -linear sheaf of L_∞ -algebras $\mathfrak{g}_{\text{red}}$ is locally quasi-isomorphic to the sheaf of sections of a graded vector bundle V , with trivial differential and L_∞ structure.
2. An L_∞ space (X, \mathfrak{g}) is *quasi-smooth* if the cohomology sheaves of $\mathfrak{g}_{\text{red}}$ are concentrated in degrees 1 and 2.
3. An L_∞ space (X, \mathfrak{g}) is *nice* if it is both locally trivial and quasi-smooth.

Considered up to equivalence, the local structure of $\mathfrak{g}_{\text{red}}$ for a nice L_∞ space (X, \mathfrak{g}) is easy to describe. Indeed by assumption (X, \mathfrak{g}) is nice and hence locally trivial, so we can assume that locally $\mathfrak{g}_{\text{red}}$ has trivial differential and L_∞ structure i.e. locally $\mathfrak{g}_{\text{red}}$ is a free C_X^∞ -module. Let d_i denote the rank of $H^i(\mathfrak{g}_{\text{red}})$. As (X, \mathfrak{g}) is additionally quasi-smooth $d_i = 0$ for $i \neq 1, 2$. Let V be the graded vector space given by

$$V = \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2}[1].$$

Locally we have an isomorphism

$$\mathfrak{g}_{\text{red}} \cong V^\vee[-1] \otimes_{\mathbb{C}} C_X^\infty.$$

The Chevalley-Eilenberg complex also has a description in terms of V :

$$C^*(\mathfrak{g}_{\text{red}}) \cong \widehat{\text{Sym}}_{C_X^\infty}((V^\vee[-1] \otimes_{\mathbb{C}} C_X^\infty)^\vee[-1]) \cong C_X^\infty \otimes_{\mathbb{C}} \widehat{\text{Sym}}_{\mathbb{C}}(V).$$

Remark 10.2. Quasi-smooth is language from algebraic geometry and a quasi-smooth scheme has a cotangent complex in which the cohomology vanishes in degrees greater than 1. Note that locally complete intersections are quasi-smooth. If a scheme is quasi-smooth then its dualizing complex is the determinant of its cotangent complex. For L_∞ spaces we should think of $\mathfrak{g}_{\text{red}}$ as morally playing the role of the shifted cotangent complex and hence the above definition.

10.1 Pretty nice spaces from nice spaces

Lemma 10.3. *If (X, \mathfrak{g}) is a nice L_∞ space, then $T^*[-1](X, \mathfrak{g}) = (X, \mathfrak{g} \oplus \mathfrak{g}^\vee[-3])$ is also nice.*

Proof. Note that as C_X^∞ -modules we have $(\mathfrak{g} \oplus \mathfrak{g}^\vee[-3])_{\text{red}} \cong (\mathfrak{g}_{\text{red}} \oplus \mathfrak{g}_{\text{red}}^\vee[-3])$. As (X, \mathfrak{g}) is locally trivial, we have a local isomorphism $\mathfrak{g}_{\text{red}} \cong V^\vee[-1] \otimes_{\mathbb{C}} C_X^\infty$ for a finite dimensional graded vector space V and consequently we have a local isomorphism

$$(\mathfrak{g} \oplus \mathfrak{g}^\vee[-3])_{\text{red}} \cong (V^\vee[-1] \oplus V[-2]) \otimes_{\mathbb{C}} C_X^\infty$$

expressing the local triviality of $T^*[-1](X, \mathfrak{g})$.

That (X, \mathfrak{g}) is nice and in particular quasi-smooth implies that V is concentrated in degree 0 and -1 . Since $(\mathfrak{g} \oplus \mathfrak{g}^\vee[-3])_{\text{red}}$ is quasi-isomorphic locally to an L_∞ algebra with no differential (or higher brackets) quasi-smoothness of $T^*[-1](X, \mathfrak{g})$ follows from observing that $V^\vee[-1] \oplus V[-2]$ (and hence the cohomology sheaves) is again concentrated in degrees 1 and 2. \square

The same technique proves the following.

Proposition 10.4.

- Let (X, \mathfrak{g}_X) denote the L_∞ space encoding the smooth geometry of X , then $T[-1](X, \mathfrak{g}_X)$ is nice.
- Let $(Y, \mathfrak{g}_{Y_\mathbb{C}})$ be the L_∞ space encoding the complex structure of a complex manifold Y (see [Cosa]), then $T[-1](Y, \mathfrak{g}_{Y_\mathbb{C}})$ is nice.

In general if (X, \mathfrak{g}) is nice, $T[-1](X, \mathfrak{g})$ is not necessarily nice as we may have non-trivial degree 3 cohomology. However, we show that we can still define integration on the L_∞ space $T[-1](X, \mathfrak{g})$.

10.2 Integrable volume forms

As we saw above, we can think of the BV laplacian as a divergence operator. That is, if ω is a volume form on $T^*[-1]X$, then the operator Δ_ω is given by Div_ω . Turning this correspondence around we can ask when a projective volume form ω on (X, \mathfrak{g}) is *integrable* and hence leads to an appropriate integral

$$\int : H^0(\mathcal{O}(X, \mathfrak{g})) \rightarrow \mathbb{C}.$$

Let ω be a projective volume form on (X, \mathfrak{g}) corresponding the quantization of $T^*[-1](X, \mathfrak{g})$ with BV laplacian Δ_ω i.e. the bracket of the BD algebra is given by

$$\{a, b\} = \Delta_\omega(a\beta) - (\Delta_\omega a)\beta - (-1)^{|a|} a(\Delta_\omega \beta).$$

Definition 10.5. The *divergence complex* associated to ω is defined by

$$\text{Div}^*(\omega) = (\mathcal{O}(T^*[-1](X, \mathfrak{g}))((\hbar)), d + \hbar \Delta_\omega).$$

We let $\mathcal{H}^i(\text{Div}^*(\omega))$ denote the i 'th cohomology sheaf of the divergence complex.

Lemma 10.6. For each i , $\mathcal{H}^i(\text{Div}^*(\omega))$ is a sheaf of $\mathbb{C}((\hbar))$ -modules and carries a \mathbb{C}^\times action lifting the action of \mathbb{C}^\times on $\mathbb{C}((\hbar))$ where \hbar has weight -1 .

The following lemma shows that $\text{Div}^*(\omega)$ is quasi-isomorphic to a local system of $\mathbb{C}((\hbar))$ lines. Later we will take \mathbb{C}^\times invariants to obtain a system of \mathbb{C} lines which in good cases, i.e. ω is *integrable*, we can identify with the orientation local system on X .

Lemma 10.7 (7.8.1 of [Cosa]). Let (X, \mathfrak{g}) be a nice L_∞ space, and let d_i denote the rank of $H^i(\mathfrak{g}_{\text{red}})$. Then, for any projective volume form ω on (X, \mathfrak{g}) , the cohomology sheaves $\mathcal{H}^i(\text{Div}^*(\omega))$ are zero except for $i = -d_1 - d_2$. Further, $\mathcal{H}^{-d_1-d_2}(\text{Div}^*(\omega))$ is a locally constant sheaf of $\mathbb{C}((\hbar))$ vector spaces of rank one.

We noted above, that in general $T[-1](X, \mathfrak{g})$ is not nice, even if (X, \mathfrak{g}) is nice. As a preliminary we need the following lemma which is a slightly more general version of the proceeding lemma.

Lemma 10.8. *Let (X, \mathfrak{g}) be a nice L_∞ space then for any projective volume form ω on $T[-1](X, \mathfrak{g})$, the cohomology sheaves $\mathcal{H}^i(\text{Div}^*(\omega))$ are zero unless $i = -2d_1$, where $d_1 = \dim H^1(\mathfrak{g}_{\text{red}})$. Further, $\mathcal{H}^{-2d_1}(\text{Div}^*(\omega))$ is a locally constant rank one sheaf of $\mathbb{C}((\hbar))$ vector spaces.*

Proof. The method of proof is the same as Costello's proof of the previous lemma. We compute locally on X the cohomology $\mathcal{O}(T^*[-1]T[-1](X, \mathfrak{g}))[[\hbar]]$ with differential $d + \hbar\Delta_\omega$. We filter $\mathcal{O}(T^*[-1]T[-1](X, \mathfrak{g}))$ by the image of multiplication by Ω_X^i . We can compute the cohomology by the associated spectral sequence and the first page is given by

$$\oplus \Omega_X^i[-i] \otimes_{C_X^\infty} C^*(\mathfrak{g}_{\text{red}}[\epsilon] \oplus \mathfrak{g}_{\text{red}}^\vee[\epsilon][-2])$$

As (X, \mathfrak{g}) is nice we can assume (since we are computing locally) that $\mathfrak{g}_{\text{red}}$ has trivial differential and L_∞ structure and is free as a C_X^∞ module. Therefore,

$$C^*(\mathfrak{g}_{\text{red}}[\epsilon] \oplus \mathfrak{g}_{\text{red}}^\vee[\epsilon][-2]) \cong C_X^\infty \otimes \widehat{\text{Sym}}(V \oplus V^\vee[1])$$

where $V = V_0 \oplus V_{-1} \oplus V_{-2}$ and the subscripts indicate in which degree the vector space lives. Let's fix a basis x_i, α_j, h_k of $V_0 \oplus V_{-1} \oplus V_{-2}$ and a dual basis x^i, α^j, h^k of $V^\vee[1]$, so the x^i have degree -1 and so on. If we let $d_i = \text{rank } \mathcal{H}^i(\mathfrak{g}_{\text{red}})$, then $\dim V_0 = d_1$, $\dim V_{-1} = d_1 + d_2$, and $\dim V_{-2} = d_2$. Then

$$\widehat{\text{Sym}}(V \oplus V^\vee[1]) = \mathbb{C}[[x_i, x^i, \alpha_j, \alpha^j, h_k, h^k]]$$

and by assumption we have a P_0 structure given by

$$\{x_i, x^i\} = 1 = \{\alpha_j, \alpha^j\} = 1 = \{h_k, h^k\}$$

with all other brackets vanishing.

Now we have a \mathbb{C}^\times equivariant quantization corresponding to ω . Such a quantization is defined by the operator Δ_ω , where

$$\Delta_\omega = \Delta_0 + \{S, -\}$$

for $S \in \mathbb{C}[[x_i]]$ and

$$\Delta_0 = \sum_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial x^i} + \sum_j \frac{\partial}{\partial \alpha_j} \frac{\partial}{\partial \alpha^j} + \sum_k \frac{\partial}{\partial h_k} \frac{\partial}{\partial h^k}.$$

We now compute the cohomology of $\mathbb{C}[[x_i, x^i, \alpha_j, \alpha^j, h_k, h^k, \hbar]]$ with respect to the differential $\hbar\Delta_0 + \hbar\{S, -\}$. We will do this via a spectral sequence associated to the filtration $F^l \mathbb{C}[[x_i, x^i, \alpha_j, \alpha^j, h_k, h^k, \hbar]]$ where the l th filtered piece consists of elements of weight greater than or equal to l . Here we give all the generators weight 1 and \hbar weight 2. Note that the differential $\hbar\Delta$ preserves this filtration.

The operator $\hbar\Delta_0$ preserves weight, while $\hbar\{S, -\}$ strictly increases weight. Hence the first page of the spectral sequence is given by

$$E_1^{p,q} = H^q(F^p \mathbb{C}[[x_i, x^i, \alpha_j, \alpha^j, h_k, h^k, \hbar]], \hbar\Delta_0).$$

In order to apply the formal Poincaré lemma we translate into the language of forms and polyvector fields. That is let ∂_{x_i} and ∂_{α^j} be of degree -1 and ∂_{h_k} be of degree 1, then we have an isomorphism

$$\mathbb{C}[[x_i, x^i, \alpha_j, \alpha^j, h_k, h^k, \hbar]] \cong \mathbb{C}[[x_i, \partial_{x_i}, \partial_{\alpha^j}, \alpha^j, h_k, \partial_{h_k}, \hbar]]$$

which sends

$$\begin{aligned} x^i &\mapsto \partial_{x_i} \\ \alpha_j &\mapsto \partial_{\alpha^j} \\ h^k &\mapsto \partial_{h_k}. \end{aligned}$$

Define the translation invariant volume form

$$dVol = dx_1 \wedge \cdots \wedge dx_{d_1} \wedge d\alpha^1 \wedge \cdots \wedge d\alpha^{d_1+d_2} \wedge dh_1 \wedge \cdots \wedge dh_{d_2}.$$

Then under the above isomorphism Δ_0 corresponds to the divergence with respect to $dVol$. Contraction with $dVol$ turns polyvector fields into forms so we have an isomorphism

$$\mathbb{C}[[x_i, x^i, \alpha_j, \alpha^j, h_k, h^k, \hbar]] \cong \mathbb{C}[[x_i, \alpha^j, h_k, dx_i, d\alpha^j, dh_k, \hbar]][2d_1]$$

which is an isomorphism of complexes after equipping the right hand side with the de Rham differential.

After inverting \hbar the Poincaré lemma shows that the cohomology is one dimensional and of degree $-2d_1$. \square

Remark 10.9. The above lemma has an analog for any L_∞ space such that $\mathfrak{g}_{\text{red}}$ has bounded cohomology.

In light of Proposition 10.4 we need to check the compatibility of the two preceding lemmas. For (X, \mathfrak{g}_X) , $H^i(\mathfrak{g}_{\text{red}})$ is zero unless $i = 1$ and in this case has rank $d_1 = \dim X$. Then

$$T[-1](X, \mathfrak{g}_X) \cong (X, \mathfrak{g}_X \oplus \mathfrak{g}_X[-1]),$$

so the cohomology of the reduced L_∞ algebra is concentrated in degrees 1 and 2 with each group being of rank $\dim X$ i.e. $d_1 = d_2 = \dim X$. Hence, we see that the lemmas concur in this case.

IN WHAT FOLLOWS WE RESTRICT TO THE CASE WHERE OUR L_∞ SPACE IS OF THE FORM $T[-1](X, \mathfrak{g})$.

As noted above, $\mathcal{H}^{-2d_1}(\text{Div}^*(\omega))$ forms a local system of $\mathbb{C}((\hbar))$ lines. By taking \mathbb{C}^\times invariants we obtain a system of \mathbb{C} lines (since \hbar has weight -1). Let

$$\mathcal{D}(\omega) = \mathcal{H}^{-2d_1}(\text{Div}^*(\omega))^{\mathbb{C}^\times}$$

denote this local system of \mathbb{C} lines. If we weight \hbar appropriately we have a quasi-isomorphism of sheaves of $\mathbb{C}((\hbar))$ -modules

$$\mathcal{D}(\omega)((\hbar))[2d_1] \cong \text{Div}^*(\omega).$$

In order to define integration, see below, we need to be able to identify $\mathcal{D}(\omega)$ with the orientation local system of X .

Definition 10.10. A projective volume form ω on $T[-1](X, \mathfrak{g})$ for (X, \mathfrak{g}) nice is *integrable* if the local system $\mathcal{D}(\omega)$ on the manifold X is isomorphic to the orientation local system on X .

10.3 The canonical volume form

Following [Cosa] we show that $T[-1](X, \mathfrak{g})$ carries a canonical volume form.

Lemma 10.11. *Let (X, \mathfrak{g}) be an L_∞ space, then we have a natural equivalence $T^*[-1]T[-1](X, \mathfrak{g}) \cong T[-1]T^*(X, \mathfrak{g})$.*

Proof.

$$\begin{aligned} T^*[-1]T[-1](X, \mathfrak{g}) &= (X, \mathfrak{g}[\epsilon] \oplus (\mathfrak{g}[\epsilon])^\vee[-3]) \\ &= (X, \mathfrak{g}[\epsilon] \oplus \mathfrak{g}^\vee[\epsilon][-2]) \\ &= (X, (\mathfrak{g} \oplus \mathfrak{g}^\vee[-2])[\epsilon]) \\ &= T[-1]T^*(X, \mathfrak{g}). \end{aligned}$$

There is some subtlety in the identifications above, we need to define the relevant L_∞ structures. The L_∞ structure on $\mathfrak{g}[\epsilon]$ is the one obtained by the product of the commutative algebra $\mathbb{C}[\epsilon]$ and L_∞ algebra \mathfrak{g} . Recall that we have the coadjoint action of \mathfrak{g} on \mathfrak{g}^\vee , we extend this ϵ -linearly to define an L_∞ structure on $\mathfrak{g}[\epsilon] \oplus \mathfrak{g}^\vee[\epsilon][-2]$. The degree -3 pairing on $(\mathfrak{g} \oplus \mathfrak{g}^\vee[-2])[\epsilon]$ is the composition of the ϵ -linear $\mathbb{C}[\epsilon]$ valued degree -2 pairing with the degree -1 map $\mathbb{C}[\epsilon] \rightarrow \mathbb{C}$ sending ϵ to 1. With these structures just defined the identifications above are actual natural equivalences of L_∞ spaces. \square

Proposition 10.12. *$T[-1](X, \mathfrak{g})$ carries a canonical volume form $d\text{Vol}_0$. That is there exists a square zero operator Δ_0 on $C^*(\mathfrak{g}[\epsilon] \oplus \mathfrak{g}^\vee[\epsilon][-2])$ such that $[d, \Delta_0] = 0$, Δ_0 has weight one with respect to the \mathbb{C}^\times action, and the failure of Δ_0 to be a derivation is the Poisson bracket.*

Proof. Let $K \in \mathfrak{g} \otimes \mathfrak{g}^\vee$ be the inverse to the canonical pairing. Define the skew-symmetric tensor

$$\tilde{K} \stackrel{\text{def}}{=} (\epsilon \otimes 1 + 1 \otimes \epsilon)K \in (\mathfrak{g}[\epsilon] \oplus \mathfrak{g}^\vee[\epsilon][-2])^{\otimes 2}.$$

Define Δ_0 to be contraction with \tilde{K} . \square

Recall that

$$C^*(\mathfrak{g}[\epsilon] \oplus \mathfrak{g}^\vee[\epsilon][-2]) = \widehat{\text{Sym}}(\mathfrak{g}[\epsilon][1] \oplus \mathfrak{g}^\vee[\epsilon][-1])^\vee$$

so Δ_0 is the unique differential operator which vanishes on constant and linear tensors and is contraction with \tilde{K} on quadratic tensors. Δ_0 can also be defined as

$$\Delta_0 = [d_{dR}, i_\Pi],$$

where Π is the Poisson tensor on $T^*(X, \mathfrak{g})$ and after identifying $\mathcal{O}(T[-1]T^*(X, \mathfrak{g}))$ with forms on $T^*(X, \mathfrak{g})$, d_{dR} denotes the de Rham differential (which lowers degree by -1).

11 $\hat{A}(B\mathfrak{g})$ as a volume form

Let (X, \mathfrak{g}) be an L_∞ space, then we can consider the one dimensional Chern-Simons theory with space of fields

$$\mathcal{L}T^*(X, \mathfrak{g}) = (X, \Omega_{S^1}^* \otimes (\mathfrak{g} \oplus \mathfrak{g}^\vee[-2])) \cong T[-1]T^*(X, \mathfrak{g}) \cong T^*[-1]T[-1](X, \mathfrak{g}).$$

We saw in Section 8.4 that we can quantize this theory. Therefore, we obtain a projective volume form on

$$\mathcal{L}(X, \mathfrak{g}) \cong T[-1](X, \mathfrak{g}).$$

Definition 11.1. Let (X, \mathfrak{g}) be an L_∞ space, then we define $dVol_{S^1}$ to be the projective volume form determined by the quantization of one dimensional Chern-Simons theory with space of fields $T[-1]T^*(X, \mathfrak{g})$. We let Δ_{S^1} denote the corresponding BV laplacian.

It follows from Theorem 8.6 that we have

$$\Delta_{S^1} = \Delta_0 + \{\log(\hat{A}(B\mathfrak{g})), -\}.$$

Lemma 11.2. Let $\Delta_\omega = \Delta_0 + \{S_\omega, -\}$ with $S_\omega \in \mathcal{O}(T[-1](X, \mathfrak{g}))$ satisfying the QME, then The map given by multiplication by $e^{S_\omega/\hbar}$

$$e^{S_\omega/\hbar} : \text{Div}^*(\omega_0) \rightarrow \text{Div}^*(\omega)$$

is a cochain isomorphism.

Proof. The lemma follows from the standard fact that

$$e^{-S_\omega} \Delta_0 e^{S_\omega} = \Delta_\omega.$$

Indeed, let $I \in \mathcal{O}(T[-1](X, \mathfrak{g}))$, then we have

$$\begin{aligned} \Delta_0(e^{S_\omega} I) &= \{e^{S_\omega}, I\} - \Delta_0(e^{S_\omega})I + e^{S_\omega} \Delta_0(I) \\ &= e^{S_\omega} \{S_\omega, I\} - e^{S_\omega} (\Delta_0(S_\omega) + 1/2\{S_\omega, S_\omega\})I + e^{S_\omega} \Delta_0(I) \\ &= e^{S_\omega} \{S_\omega, I\} + e^{S_\omega} \Delta_0(I), \end{aligned}$$

where the last equality follows from S_ω satisfying the QME. □

We obtain as an immediate corollary of the lemma the following.

Proposition 11.3. Let (X, \mathfrak{g}) be an L_∞ space and let $dVol_0$ be the canonical projective volume form on $T[-1](X, \mathfrak{g})$. Further, let $dVol_{S^1}$ be the projective volume form determined by the one loop quantization of one dimensional Chern-Simons theory. Then $dVol_{S^1}$ is integrable if and only if $dVol_0$ is integrable.

So far we have not found a general criterion for the integrability of $dVol_0$ and hence $dVol_{S^1}$. If our L_∞ space is $(X, \mathfrak{g}_{X_{\bar{\partial}}})$ for a complex manifold X , then $dVol_0$ is integrable. The case of (X, \mathfrak{g}_X) for X smooth is more subtle as we discuss below.

12 Integration

Suppose that (X, \mathfrak{g}) is a nice L_∞ space and that ω is an integrable projective volume form on $T[-1](X, \mathfrak{g})$. By definition the projection map of the shifted cotangent bundle

$$\mathfrak{g}[\epsilon] \oplus (\mathfrak{g}[\epsilon])^\vee[-3] \rightarrow \mathfrak{g}[\epsilon]$$

is a map of L_∞ algebras and hence induces a natural pull back map on functions

$$\mathcal{O}(T[-1](X, \mathfrak{g})) \rightarrow \mathcal{O}(T^*[-1]T[-1](X, \mathfrak{g}))$$

which leads to a map of sheaves

$$\mathcal{O}(T[-1](X, \mathfrak{g})) \rightarrow \text{Div}^*(\omega).$$

We pass to compactly supported cohomology to obtain a map

$$H_c^i(X, \mathcal{O}(T[-1](X, \mathfrak{g}))) \rightarrow H_c^i(X, \text{Div}^*(\omega))$$

and taking \mathbb{C}^\times invariants gives

$$H_c^i(X, \mathcal{O}(T[-1](X, \mathfrak{g}))) \rightarrow H_c^i(X, (\text{Div}^*(\omega))^{\mathbb{C}^\times}) \cong H_c^{i+2d_1}(X, \mathcal{D}(\omega)).$$

Now as $\mathcal{D}(\omega)$ is isomorphic to the orientation local system on X , so the right hand side is zero unless $i + 2d_1$ is the dimension of X and in this dimension it is one-dimensional.

Definition 12.1. The integral associated to an integrable volume form ω on $T[-1](X, \mathfrak{g})$, with $\dim X = n$, is the map

$$H_c^{n-2d_1}(X, \mathcal{O}(T[-1](X, \mathfrak{g}))) \rightarrow H_c^n(X, \mathcal{D}(\omega)) \cong \mathbb{C}.$$

12.1 Integration on $T[-1](X, \mathfrak{g})$

We begin by recalling Costello's work with complex manifolds as the theorems have a particularly nice form in this setting. To begin, Costello shows that for $(X, \mathfrak{g}_{X_{\overline{\partial}}})$ the L_∞ space encoding the complex structure of X (real dimension of X is $2n$), then

$$H^0(X, \mathcal{O}(T[-1](X, \mathfrak{g}_{X_{\overline{\partial}}})) = \oplus H^i(X, \Omega_{X, \text{hol}}^n).$$

Theorem 12.2 (Theorem 8.0.3 of [Cosa]). *The projective volume form $d\text{Vol}_0$ on $T[-1](X, \mathfrak{g}_{X_{\overline{\partial}}})$ is integrable and the associated integral*

$$\int : H^0(X, \mathcal{O}(T[-1](X, \mathfrak{g}_{X_{\overline{\partial}}})) \rightarrow \mathbb{C}$$

is (up to a scalar) the usual integration on $H^n(X, \Omega_{X, \text{hol}}^n)$ and vanishes on $H^i(X, \Omega_{X, \text{hol}}^i)$ for $i < n$.

Costello quantizes the field theory given by maps $E \rightarrow T^*X$ (where E is an elliptic curve with fixed volume form) which is called *holomorphic Chern-Simons* and denotes the resulting projective volume form by $d\text{Vol}_E$.

Theorem 12.3 (Main theorem of [Cosa]). *The quantization of holomorphic Chern-Simons theory with target a complex manifold X determines an integrable projective volume form $dVol_E$ on $T[-1](X, \mathfrak{g}_{X_{\bar{\partial}}})$ with associated integral*

$$\int_{T[-1](X, \mathfrak{g}_{X_{\bar{\partial}}})} \alpha dVol_E = \int_{T[-1](X, \mathfrak{g}_{X_{\bar{\partial}}})} \alpha \cdot \text{Wit}(X, E) dVol_0,$$

where $\text{Wit}(X, E) \in \oplus H^i(X, \Omega_X^i)$ is the Witten class.

From Theorem 8.8 we know that one dimensional Chern-Simons also defines a projective volume form on $T[-1](X, \mathfrak{g}_{X_{\bar{\partial}}})$ which we will denote \overline{dVol}_{S^1} . That this volume form is integrable follows from the general results proved by in [Cosa]. We then have the following.

Theorem 12.4. *The quantization of one dimensional Chern-Simons theory with target a complex manifold X determines an integrable projective volume form \overline{dVol}_{S^1} on $T[-1](X, \mathfrak{g}_{X_{\bar{\partial}}})$ with associated integral*

$$\int_{T[-1](X, \mathfrak{g}_{X_{\bar{\partial}}})} \alpha \overline{dVol}_{S^1} = \int_{T[-1](X, \mathfrak{g}_{X_{\bar{\partial}}})} \alpha \cdot e^{-c_1(X)/2} Td(X) dVol_0.$$

For a general L_∞ space we also obtain a volume form $dVol_{S^1}$ on $T[-1](X, \mathfrak{g})$ from the quantization of one dimensional Chern-Simons theory (this is the one from the previous section). However, as we noted above we do not know a general integrability criterion. In the case that $dVol_0$ and $dVol_{S^1}$ are both integrable we have the following.

Theorem 12.5. *Let (X, \mathfrak{g}) be an L_∞ space such that $dVol_0$ is integrable, then*

$$\int_{T[-1](X, \mathfrak{g})} \alpha dVol_{S^1} = \int_{T[-1](X, \mathfrak{g})} \alpha \cdot \hat{A}(B\mathfrak{g}) dVol_0.$$

The theorem follows from a simple lemma.

Lemma 12.6. *Let ω_0 and ω be integrable volume forms on an L_∞ space (X, \mathfrak{g}) such that for the corresponding BV laplacians we have*

$$\Delta_\omega = \Delta_0 + \{S_\omega, -\}.$$

Then for $\alpha \in H^0(X, \mathcal{O}(X, \mathfrak{g}))$ we have

$$\int_{(X, \mathfrak{g})} \alpha \omega = \int_{(X, \mathfrak{g})} \alpha e^{S_\omega} \omega_0.$$

Proof. We know from Lemma 11.2 that multiplication by $e^{S_\omega/\hbar}$ is a cochain isomorphism between divergence complexes, hence we have a commutative diagram

$$\begin{array}{ccc} & H_c^i(X, \text{Div}^*(\omega_0)^{\mathbb{C}^\times}) & \\ \nearrow \int \omega_0 & \downarrow \cong e^{S_\omega/\hbar} & \\ H_c^i(X, \mathcal{O}(X, \mathfrak{g})) & & \\ \searrow \int \omega & \downarrow & \\ & H_c^i(X, \text{Div}^*(\omega)^{\mathbb{C}^\times}) & \end{array}$$

□

12.2 The smooth case

In this final section we discuss integration on $T[-1](X, \mathfrak{g}_X)$ for X a smooth, compact, oriented manifold. We will see that $dVol_0$ and hence $dVol_{S^1}$ are integrable, but the associated integral is trivial. We then take homotopy invariants with respect to the action of $B\mathbb{G}_a$ in order to obtain a potentially non trivial integration map.

Proposition 12.7. *The canonical volume form $dVol_0$ on $T[-1](X, \mathfrak{g}_X)$ is integrable.*

Proof. We can identify the divergence complex $\text{Div}^*(dVol_0)$ with $\Omega_{T^*B\mathfrak{g}}^{-*}$ which is quasi-isomorphic to $dR(J(\Omega_{T^*X}^{-*}))(\hbar)$ which itself is quasi-isomorphic to $\Omega_{T^*X}^{-*}(\hbar)$. Hence the cohomology sheaves of this complex are zero except in dimension $-2n$, where $\dim X = n$, in which they are $\mathbb{C}(\hbar)$. We see that $\mathcal{D}(dVol_0)$ is the trivial local system and we assume X is oriented and hence the proposition follows. \square

Note that we know immediately that since $dVol_0$ is integrable, so is $dVol_{S^1}$.

The integral associated to $dVol_0$ on $T[-1](X, \mathfrak{g}_X)$ is a map

$$\int : H^{-n}(\mathcal{O}(T[-1](X, \mathfrak{g}_X))) \rightarrow H^n(X, \mathcal{D}(dVol_0)) \cong \mathbb{C}.$$

But we have

$$H^{-n}(\mathcal{O}(T[-1](X, \mathfrak{g}_X))) = 0.$$

In order to have an interesting integral we must remember the action of $B\mathbb{G}_a$. So far we have constructed a map of degree 0 map of sheaves

$$\int : \mathcal{O}(T[-1](X, \mathfrak{g}_X)) \rightarrow \mathbb{C}[n].$$

We now pass to $B\mathbb{G}_a$ (homotopy) invariants where we equip \mathbb{C} with the trivial $B\mathbb{G}_a$ module structure. Therefore, we obtain a map

$$\int dVol_0^{B\mathbb{G}_a} : (\mathcal{O}(T[-1](X, \mathfrak{g}_X)))^{B\mathbb{G}_a} \simeq (\Omega_X^{-*}[[u]], ud) \rightarrow \mathbb{C}[[u]][n].$$

It is still work in progress to identify this integral. The difficulty is in writing down an equivariant version of the divergence complex. If this integral were well understood then using Theorem 12.5 would could obtain a map obtained by integrating with respect to $dVol_{S^1}^{B\mathbb{G}_a}$. One would hope that integrating the constant function 1 against $dVol_{S^1}^{B\mathbb{G}_a}$ evaluates to $\hat{A}(X)$ (possibly with a factor of u^n).

A Algebraic preliminaries

A.1 Graded linear algebra

Let us recall some graded linear algebra. Let R denote a ground ring (in practice it will be an algebra over \mathbb{C} or \mathbb{R}) and M a \mathbb{Z} -graded module over R . The tensor algebra $T_R(M)$ is then \mathbb{Z} -graded with

$$T_R(M)^i = \bigoplus_{j_1 + \dots + j_k = i} M_{j_1} \otimes \dots \otimes M_{j_k}.$$

There is an action of the symmetric group S_n on $M^{\otimes n}$ given by

$$\sigma(m_1 \otimes \cdots \otimes m_n) = \epsilon(\sigma, m_1, \dots, m_n) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)},$$

with

$$\epsilon(\sigma, m_1, \dots, m_n) = \prod_{i < j} \left(\frac{\sigma(i) - \sigma(j)}{|\sigma(i) - \sigma(j)|} \right)^{|m_i| \cdot |m_j|}.$$

So in particular if $\sigma \in S_2$ is non trivial then

$$\epsilon(\sigma, m_1, m_2) = (-1)^{|m_1| \cdot |m_2|}.$$

We define the n th symmetric power by

$$\text{Sym}_R^n(M) \stackrel{\text{def}}{=} (M^{\otimes n})_{S_n},$$

where the subscript denotes taking coinvariants with respect to the S_n action defined above. We define the completed symmetric algebra as the direct product

$$\widehat{\text{Sym}}_R(M) \stackrel{\text{def}}{=} \prod_{n \geq 0} \text{Sym}_R^n(M).$$

We define the n th exterior power of M by

$$\Lambda_R^n(M) \stackrel{\text{def}}{=} \text{Sym}_R^n(M[1])[-n].$$

One can check that if M is concentrated in degree 0, this definition concurs with the conventional definition. Our shifting convention is that elements of degree n in M are of degree $n - k$ in $M[k]$.

Also note that given two graded R -modules the set of homomorphisms acquires the natural structure of a graded R -module.

Finally, suppose I have a \mathbb{R} vector space V and a homogeneous map of degree k $f : V \rightarrow \mathbb{R}$, so $f(tv) = t^k f(v)$, then I can use *polarization* to obtain a S_k invariant, k multilinear map $D^k f : V^{\otimes k} \rightarrow \mathbb{R}$. Indeed, let $v_1 \otimes \dots \otimes v_k \in V^{\otimes k}$ then $D^k f$ is the assignment

$$v_1 \otimes \dots \otimes v_k \mapsto \left(\frac{\partial}{\partial v_1} \cdots \frac{\partial}{\partial v_k} f \right) (0).$$

A.2 Nuclear spaces

In later considerations we will be using topological vector spaces, so we recall some basic notions here. A standard reference is Trèves [Trè67] though we mostly follow Appendix 2 of [Cos11]. We will be mostly concerned with nuclear Fréchet spaces, this allows us to perform most of the algebraic constructions described in the previous section.

A *topological vector space* over \mathbb{R} or \mathbb{C} is a vector space with a topology such that scalar multiplication and addition are continuous. We will be interested in topological vector spaces that are Hausdorff and locally convex (so there is a basis for the topology made up of convex sets). Any

locally convex space has a family of seminorms which determine the topology and if this family is countable (and the space is Hausdorff) one can complete and the resulting space is Fréchet. Equivalently, a Fréchet space is a complete, metrizable locally convex space.

Given a topological vector space V we let V^\vee denote the space of continuous linear functionals and equip V^\vee with the strong topology (so that of bounded uniform convergence). There are a whole slew of possible tensor products on locally convex vector spaces and we are primarily interested in the *completed projective tensor product*, that is the completion of the algebraic tensor product $V \otimes_{alg} W$ with respect to the projective topology (which is the finest topology so that the map $V \times W \rightarrow V \otimes_{alg} W$ is continuous). There is similarly a completed injective tensor product (see [Trè67]).

Throughout, unless otherwise noted, we use the completed tensor product.

There are many equivalent definitions of *nuclear space* but one is to say that a locally convex Hausdorff space V is nuclear if for any other locally convex Hausdorff space W the map from the completed projective tensor product to the completed injective tensor product is an isomorphism. Nuclear spaces form a symmetric monoidal category where morphisms are continuous linear maps and the monoidal structure is the completed projective tensor product. This category is not enriched over itself, but is enriched in ordinary vector spaces. The category of nuclear spaces has all limits and limits commute with the monoidal structure. Many spaces from functional analysis are actually nuclear including smooth functions on open sets of \mathbb{R}^n , compactly supported smooth functions on such an open set, and the dual spaces of distributions. Similarly, smooth functions and their dual space of distributions on a manifold M are nuclear (actually nuclear Fréchet). The projective tensor product is such that for two manifolds M and N we have

$$C_M^\infty \otimes C_N^\infty = C_{M \times N}^\infty$$

and the same holds for distributions or sections of vector bundles.

We can import most of the linear algebra recalled in the previous section to the category of nuclear spaces, actually we restrict to nuclear Fréchet spaces. Let E be a nuclear Fréchet space, then we define

$$\text{Sym}^n(E) \stackrel{\text{def}}{=} (E^{\otimes n})_{S_n}.$$

We similarly can define the algebra of formal power series on E as a nuclear space as

$$\mathcal{O}(E) \stackrel{\text{def}}{=} \prod_{n \geq 0} \text{Sym}^n(E^\vee).$$

This construction defines a commutative algebra in nuclear spaces and is compatible with base change in a sense we now describe. Let A be a commutative algebra in nuclear spaces, then we have

$$\mathcal{O}(E) \otimes A = \prod_{n \geq 0} \text{Sym}^n(E^\vee) \otimes A = \prod_{n \geq 0} \text{Hom}(E^{\otimes n}, A)_{S_n}$$

where $\text{Hom}(E^{\otimes n}, A)$ is again equipped with the strong topology. In this identification we use the fact that for E nuclear Fréchet and F any nuclear space we have

$$\text{Hom}(E, F) = E^\vee \otimes F.$$

A.3 Derivations

Here we recall the existence of Kähler differentials for dg algebras. We follow [Man].

Let (R, d_R) be a differential graded algebra (over some base ring K) and (M, d_M) a dg R -module, then there exists a dg R -module of derivations given by

$$\mathrm{Der}(R, M) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}^n(R, M),$$

where

$$\mathrm{Der}^n(R, M) = \{D \in \mathrm{Hom}_K^n(R, M) : D(ab) = D(a)b + (-1)^{n|a|}aD(b)\}.$$

The differential $d : \mathrm{Der}^n(R, M) \rightarrow \mathrm{Der}^{n+1}(R, M)$ is given by

$$d\phi = d_M \circ D - (-1)^n D \circ d_R.$$

Proposition A.1. *There exists a dg R -module Ω_R^1 and a square zero degree 0 derivation $\delta : R \rightarrow \Omega_R^1$ such that for every dg R -module M , composition with δ gives an isomorphism*

$$\mathrm{Hom}_R(\Omega_R^1, M) \cong \mathrm{Der}(R, M).$$

Sketch of proof. Define the graded vector space

$$F_R = \bigoplus R\delta x, \quad x \in R \text{ homogenous.}$$

F_R is a dg R -module via

$$a(b\delta x) = ab\delta x \text{ and } d_\Omega(a\delta x) = d_R a\delta x + (-1)^{|a|}a\delta(d_R x).$$

In particular, we have $d_\Omega(\delta x) = \delta(d_R x)$. Now let I be the submodule generated by

$$\delta(x+y) - \delta x - \delta y, \quad \delta(xy) - x\delta(y) - (-1)^{|x||y|}y\delta(x), \quad \delta(k) = 0,$$

for any $k \in K$ the base ring. Note that $d_\Omega(I) \subset I$, so can define

$$\Omega_R^1 = F_R / I$$

which is still a dg R -module. $\delta : R \rightarrow \Omega_R^1$ is a derivation such that

$$d\delta = d_\Omega\delta - \delta d_R = 0,$$

where d is the differential on derivations. One can check that

$$\circ\delta : \mathrm{Hom}_R(\Omega_R^1, M) \rightarrow \mathrm{Der}(R, M)$$

is an isomorphism. □

The dg R -module Ω_R^1 is the module of Kähler differentials and we let $d_{dR} = \delta : R \rightarrow \Omega_R^1$ denote the universal derivation (we have avoided this notation in the proposition as an attempt to aid in clarity). Per the standard universality, Ω_R^1 is unique up to isomorphism.

Note that if R^\sharp (that is the underlying graded algebra) is a polynomial algebra $K[x_1, \dots]$, then $\Omega_R = \bigoplus R x_i$ and d_{dR} is the unique derivation such that $d_{dR}(x_i) = \delta x_i$.

Further, one can construct relative Kähler differentials and show the standard facts about base change, etc., but we won't need this machinery in the sequel.

B D -modules

For X a smooth manifold, let D_X denote the ring of smooth differential operators on X . There are many ways to define this ring. For instance, D_X is the subalgebra of $\text{End}_C(C_X^\infty, C_X^\infty)$ generated by left multiplication by C_X^∞ and by smooth vector fields T_X . Locally, every differential operator P has the form

$$P = \sum_{\alpha} a_{\alpha}(x) \partial^{\alpha},$$

where the a_{α} are smooth functions and ∂^{α} is the multinomial notation for a partial derivative.

A left D_X module M is simply a left module for this algebra. One natural source of left D_X modules is given by smooth vector bundles with flat connections. Let E be a smooth vector bundle over X and let \mathcal{E} denote its smooth sections. If \mathcal{E} is a left D_X module (extending the action of C_X^∞), then every vector field acts on \mathcal{E} : we have $X \cdot s \in \mathcal{E}$ for every vector field $X \in T_X$ and every smooth section $s \in \mathcal{E}$. Equipping \mathcal{E} with an action of vector fields is equivalent to putting a connection ∇ on E . Moreover, we have $[X, Y] \cdot s = X \cdot (Y \cdot s) - Y \cdot (X \cdot s)$ for all $X, Y \in T_X$ and $s \in \mathcal{E}$. To satisfy the bracket relation, this connection ∇ must be flat.

There is a forgetful functor $F : D_X\text{-mod} \rightarrow C_X^\infty\text{-mod}$, where we simply forget about how vector fields act on sections of the sheaf. As usual, there is a left adjoint to F given by tensoring with D_X :

$$D_X \otimes_{C_X^\infty} - : M \mapsto D_X \otimes_{C_X^\infty} M.$$

Using the forgetful functor, we can equip the category of left D_X modules with a symmetric monoidal product. Namely, we tensor over C_X^∞ and equip $M \otimes_{C_X^\infty} N$ with the natural D_X structure

$$X \cdot (m \otimes n) = (X \cdot m) \otimes n + m \otimes (X \cdot n),$$

for any $X \in T_X$, $m \in M$, and $n \in N$. By construction, C_X^∞ is the unit object in the symmetric monoidal category of left D_X modules. We will write $M \otimes N$ to denote $M \otimes_{C_X^\infty} N$ unless there is a possibility of confusion.

Remark B.1. Right D_X modules also appear in this paper and throughout mathematics. For instance, distributions and the sheaf of densities Dens_X are naturally right D_X modules, since distributions and densities pair with functions to give numbers. Since we are working with smooth manifolds, however, it is easy to pass back and forth between left and right D_X modules.

C Jets

There is another, beautiful way to relate vector bundles and D_X modules, and we will use it extensively in our constructions. Given a finite rank vector bundle E on X , the infinite jet bundle $J(E)$ is naturally a D_X module, as follows. Recall that for a smooth function f , the ∞ -jet of f at a point $x \in X$ is its Taylor series (or, rather, the coordinate-independent object that corresponds to a Taylor series after giving local coordinates around x). We can likewise define the ∞ -jet of a section s of E at a point x . The bundle $J(E)$ is the infinite-dimensional vector bundle whose fiber at a point x is the space of ∞ -jets of sections of E at x . This bundle has a tautological connection,

since knowing the Taylor series of a section at a point automatically tells us how to do infinitesimal parallel transport. Nonetheless, it is useful to give an explicit formula. Let x be a point in X and pick local coordinates x_1, \dots, x_n in a small open neighborhood U of x . Pick a trivialization of E over U so that

$$\Gamma(U, J(E)) \cong C^\infty(U) \otimes_{\mathbb{R}} \mathbb{R}[[x_1, \dots, x_n]] \otimes_{\mathbb{R}} E_x.$$

We write a monomial $x_1^{a_1} \cdots x_n^{a_n}$ using multinomial notation: for $\alpha = (a_1, \dots, a_n) \in \mathbb{N}^n$, x^α denotes the obvious monomial. Hence, given a section $f \otimes x^\alpha \otimes e \in C^\infty(U) \otimes_{\mathbb{R}} \mathbb{R}[[x_1, \dots, x_n]] \otimes_{\mathbb{R}} E_x$ and vector field $\partial_j = \partial/\partial x_j$, the connection is

$$\partial_j \cdot f \otimes x^\alpha \otimes e = (\partial_j f) \otimes x^\alpha \otimes e - f \otimes (\alpha_j x^{\alpha - e_j}) \otimes e.$$

We are just applying the vector field in the natural way first to the function and then to the monomial. It is a simple computation that this defines a flat connection. As we will see below the horizontal sections of $J(E)$ are the smooth sections of E .

What makes this construction useful is that it allows one to translate questions about geometry into questions about D_X modules. There is a rich literature explaining how to exploit this translation, and the usual name for this area of mathematics is (Gelfand-Kazhdan) *formal geometry*.

There is another way to construct the sheaf of sections of $J(E)$. Let \mathcal{J} denote the sheaf of sections of J , the jet bundle for the trivial rank 1 bundle over X . Observe for any point $p \in X$,

$$\mathcal{J}_p = \varprojlim C_X^\infty / \mathfrak{m}_p^k,$$

where \mathfrak{m}_p denotes the maximal ideal of functions vanishing at p . This equips \mathcal{J} with a canonical filtration by “order of vanishing.” The equivalence of these local descriptions essentially is a consequence of the chain rule (see [KMS93]). Now let \mathcal{E} denote the sheaf of smooth sections of E , which is a module over C_X^∞ . Then the sheaf $\mathcal{J}(E)$ has stalk

$$\mathcal{J}(E)_p = \varprojlim \mathcal{E} / \mathfrak{m}_p^k \mathcal{E},$$

and hence also has a natural filtration by order of vanishing. Moreover, this shows that $J(E)$ is a module over \mathcal{J} . We will use the following lemma repeatedly in our constructions.

Lemma C.1. *A C_X^∞ splitting $\sigma : \mathcal{E} \rightarrow \mathcal{J}(E)$ of the canonical quotient map $q : \mathcal{J}(E) \rightarrow \mathcal{E}$ induces an isomorphism $i_\sigma : \mathcal{J}(E) \cong \mathcal{E} \otimes_{C_X^\infty} \mathcal{J}$ as \mathcal{J} -modules.*

Proof. Observe that $\mathcal{J}(E)$ is a \mathcal{J} -module just as \mathcal{E} is a C_X^∞ -module. Thus we obtain a map

$$\mathcal{J} \otimes_{C_X^\infty} \mathcal{E} \rightarrow J(E)$$

$$j \otimes s \mapsto j \cdot \sigma(s).$$

We need to show this map is an isomorphism of C_X^∞ modules. It is enough to check this locally, so notice that for any small ball $B \subset X$, if we pick coordinates x_1, \dots, x_n on B , we get trivializations

$$E|_B \cong C_X^\infty(B) \otimes E_0, J|_B \cong C_X^\infty(B) \otimes \mathbb{R}[[x_1, \dots, x_n]], \text{ and}$$

$$J(E)|_B \cong C_X^\infty(B) \otimes \mathbb{R}[[x_1, \dots, x_n]] \otimes E_0,$$

where E_0 denotes the fiber of E over the point $0 \in B$. Let $\{e_i\}$ denote a basis for E_0 ; the “constant” sections $\{1 \otimes e_i\}$ in \mathcal{E} then form a frame for \mathcal{E} over B . Let $s_i = \sigma(e_i)$. Notice that under the map $\mathcal{J}(E)/F^1 \rightarrow \mathcal{E}$, s_i goes to e_i , and so the s_i are linearly independent in $\mathcal{J}(E)$. By linear algebra over \mathcal{J} , one obtains that the map i_σ is an isomorphism. \square

D Ω -modules

Let Ω_X denote the de Rham complex of X and $\Omega_X^\#$ the underlying graded algebra. An Ω_X module is a graded module M^* over $\Omega_X^\#$ with a differential ∂ that satisfies

$$\partial(\omega \cdot m) = (d\omega) \cdot m + (-1)^{|\omega|} \omega \cdot \partial m,$$

where $\omega \in \Omega_X$ and $m \in M$. A natural source of examples is (again!) vector bundles with flat connection. Let E be a vector bundle. Differential forms with values in E , $\Omega_X^\#(E)$, naturally form a graded module over $\Omega_X^\#$. Equipping $\Omega_X^\#(E)$ with a differential is exactly the same data as a flat connection ∇ on E . We call it the *de Rham complex* of (E, ∇) and will denote it $dR(E)$. Explicitly, we have

$$\begin{aligned} \partial : \Omega_X^* \otimes_{C_X^\infty} \mathcal{E} &\rightarrow \Omega_X^{*+1} \otimes_{C_X^\infty} \mathcal{E} \\ \omega \otimes e &\mapsto d\omega \otimes e + (-1)^{|\omega|} \sum_i dx_i \wedge \omega \otimes \nabla_{\partial/\partial x_i} e. \end{aligned}$$

Note that $\partial^2 = 0$ follows from the flatness of ∇ .

Lemma D.1. *Let (E, ∇) be a vector bundle with flat connection. Then the horizontal sections of E is isomorphic to zero cohomology of the associated de Rham complex, i.e.*

$$\ker \nabla \cong H^0(dR(E)).$$

Proof. Because the construction is C_X^∞ linear, it is enough to consider an element of the form $1 \otimes e \in \Omega_X^0 \otimes_{C_X^\infty} \mathcal{E}$. The lemma is then obvious as applying the differential of the de Rham complex yields

$$\begin{aligned} \partial(1 \otimes e) &= d(1) \otimes e + \sum dx_i \otimes \nabla_{\partial/\partial x_i} e \\ &= \sum dx_i \otimes \nabla_{\partial/\partial x_i} e \end{aligned}$$

So $\partial(1 \otimes e) = 0$ if and only if $e \in \mathcal{E}$ is a flat section. \square

Consider the jet bundle J with its flat connection then we see that the cohomology is concentrated in degree 0; we defer to [CFT02] (Lemma 4.7) for the proof.

Lemma D.2. *The cohomology of the de Rham complex of (J, ∇) is concentrated in degree 0, that is*

$$H^p(dR(\mathcal{J})) = \begin{cases} C_X^\infty, & p = 0, \\ 0, & p > 0. \end{cases}$$

The category of Ω_X modules is symmetric monoidal in the obvious way. Given two Ω_X modules M and N , then $M \otimes_{\Omega_X} N$ is, as a graded module, the tensor product $M \otimes_{\Omega_X} N$ equipped with differential

$$\partial(m \otimes n) = \partial_M m \otimes n + (-1)^{|m|} m \otimes \partial_N n.$$

Of course, it is better to work with the derived tensor product in most situations.

Since Ω_X is commutative, there is a dg manifold $X_\Omega = (X, \Omega_X)$. It clearly captures the smooth topology of the manifold X . Many of our constructions in this paper involve X_Ω . Moreover, many classical constructions in differential geometry (e.g., the Frölicher-Nijenhuis bracket) appear most naturally as living on X_Ω .

E The de Rham complex of a left D -module

Earlier, we explained how a vector bundle with flat connection (E, ∇) is a left D -module and how to use the connection to make $\Omega^*(E)$ into an Ω_X module. We now extend this construction to all left D -modules.

Let M be a left D -module. The *de Rham complex* $dR(M)$ of M (or the *de Rham functor applied to* M) consists of the graded C_X^∞ -module $\Omega_X^\# \otimes_{C_X^\infty} M$ equipped with the differential

$$d_M : \omega \otimes m \mapsto d\omega \otimes m + (-1)^{|\omega|} \sum_i dx_i \wedge \omega \otimes \frac{\partial}{\partial x_i} m.$$

By construction, $dR(M)$ is an Ω_X -module.

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